

# Repteori - Week 4.

$\nu^* = \text{Hom}_A(D_-, A)$  is the opposite order of  $\nu$ .

Notice that  $\nu^* \cong \text{Hom}_A(DA, -)$ , and is right adjoint to  $\nu: \text{mod}_A \rightarrow \text{mod}_A$  s.t.  $\nu$  restricts to an equivalence  $\nu \text{proj}_A: \text{proj}_A \rightarrow \text{inj}_A$ .

## Construction:

$M \in \text{mod } A$ , Pick a minimal projective presentation:

$$P_1 \xrightarrow{p} P_0 \rightarrow M \rightarrow 0. \text{ Define } \tau \text{ as } \ker \nu p, \text{ i.e.}$$

$$0 \rightarrow \tau M \rightarrow \nu P_1 \xrightarrow{\nu p} \nu P_0.$$

$\text{Tr}$  is called the trace

Remark:  $\tau$  is also known as  $D\text{Tr}$

$\text{Tr} M$  is defined as the cokernel of  $\text{Hom}(p, A)$ .

Another remark: We can do the inverse construction to get  $\tau^-$  from  $\nu^-$ . Pick an injective copresentation, and define  $\tau^-$  as  $\text{coker } \nu^- i$ :

$$0 \rightarrow M \rightarrow I^0 \xrightarrow{i} I^1$$

$$\downarrow \nu^-$$

$$\nu^- I^0 \xrightarrow{\nu^- i} \nu^- I^1 \rightarrow \tau^- M \rightarrow 0.$$

Ex.  $A = K[x \rightarrow y]$ ,  $M = [0 \leftarrow K]$ , Construct  $\tau$  of  $M$ .

$$\{K \leftarrow 0\} \xrightarrow{x^0} \{K \leftarrow K\} \rightarrow M \rightarrow 0$$

$$\downarrow \nu$$

$$0 \rightarrow \{K \leftarrow 0\} \rightarrow \{K \leftarrow K\} \rightarrow \{0 \leftarrow K\}$$

$A$  f.d. and  $P$  proj.

$$0 \rightarrow P \rightarrow P \rightarrow 0$$

$\downarrow \nu$

$$0 \rightarrow 0 \rightarrow 0 \rightarrow \nu P \rightarrow \tau P = 0.$$

Def - Projective stable  
 The stable module category  $\underline{\text{mod}}^A$  is given by:

$\rightarrow \text{Obj}(\underline{\text{mod}}^A) = \text{Obj}(\text{mod}^A)$

$\rightarrow \underline{\text{Hom}}_A(M, N) = \text{Hom}_A(M, N) / \text{maps factoring through a projective.}$

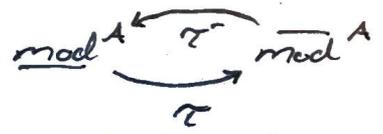
If  $P$  is projective, then  $P \approx 0$  in  $\underline{\text{mod}}^A$ .

Dually,  $\overline{\text{mod}}^A$  is given by: ← Injective stable (costable?)

$\rightarrow \text{Obj}(\overline{\text{mod}}^A) = \text{Obj}(\text{mod}^A)$

$\rightarrow \overline{\text{Hom}}_A(M, N) = \text{Hom}_A(M, N) / \text{maps factoring through an injective}$

Thm \*  $\tau$  and  $\tau^{-1}$  induce mutually inverse equivalences:

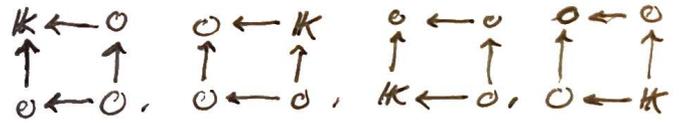


~ Start of lecture ~

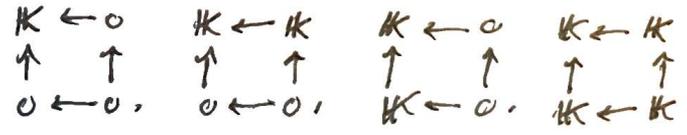
Calculating projectives and injectives:

\*  $\mathbb{K} \begin{bmatrix} 1 & \xrightarrow{a} & 2 \\ \downarrow c & & \downarrow d \\ 3 & \xrightarrow{a} & 4 \end{bmatrix}$  (kba-de)

Simples:



indee. Proj:



indee. inj:



Continuity from thm \*

$\tau$  and  $\tau^{-1}$  are quasi-inverses if well-defined, as they undo each other by construction. Well-definedness is however the only obstacle.

Pick a morphism  $f: M \rightarrow N$ .

$$\begin{array}{ccccccc} P_1 & \rightarrow & P_0 & \rightarrow & M & \rightarrow & 0 \\ \downarrow \exists f_1 & & \downarrow \exists f_0 & & \downarrow f & & \\ Q_1 & \rightarrow & Q_0 & \rightarrow & N & \rightarrow & 0 \end{array}$$

Maps between  $P_i \rightarrow Q_i$  obtained from projective resolution property. See homalg notes for more info.

Apply  $\nu$  to diagram:

$$\begin{array}{ccccccc} 0 & \rightarrow & \tau M & \xrightarrow{\tau f} & \nu P_1 & \rightarrow & \nu P_0 \\ \text{kernel} & & & & \downarrow \nu f_1 & & \downarrow \nu f_0 \\ \text{Prop} & & & & & & \\ 0 & \rightarrow & \tau N & \xrightarrow{\tau f} & \nu Q_1 & \rightarrow & \nu Q_0 \end{array}$$

Problem:  $f_0, f_1$  not unique, so  $\nu f_i$  not unique either.  $\tau f$  is called our candidate.

Subtracting different choices for the same  $f$ . Can assume  $f=0$ , need to show  $\tau f=0$ , indep. choices.

$f=0$  in  $\text{mod } A \Leftrightarrow f$  factors through  $Q_0 \rightarrow N$  by  $h$ .

$f$  factors through projective.

$$\begin{array}{ccccc} P_1 & \xrightarrow{f_1} & P_0 & \xrightarrow{f_0} & M \\ \downarrow f_1 & \swarrow \ell & \downarrow f_0 & \swarrow h & \downarrow \\ Q_1 & \xrightarrow{q_1} & Q_0 & \xrightarrow{q_0} & N \end{array}$$

$$q_0 \circ f_0 = f_0 \circ p_0 = q_0 \circ h \circ p_0$$

$\Rightarrow f_0 - h \circ p_0$  factors through  $\text{Ker } q_0$ .

As  $P_0$  proj: even factors through proj cover of kernel, i.e. through  $q_1$  by  $l$ .

$$\begin{array}{ccccccc} \tau M & \xrightarrow{\tau f} & \nu P_1 & \xrightarrow{\nu f_1} & \nu P_0 \\ \downarrow \tau f & \swarrow m & \downarrow \nu f_1 & \swarrow \nu \ell & \downarrow \nu f_0 \\ \tau N & \xrightarrow{\tau f} & \nu Q_1 & \xrightarrow{\nu q_1} & \nu Q_0 \end{array}$$

$$\text{In particular, } f_0 \circ p_1 = q_1 \circ l \circ p_1$$

$$\Rightarrow \nu f_0 \circ \nu p_1 = \nu q_1 \circ \nu l \circ \nu p_1 = \nu q_1 \circ \nu f_1$$

$$\text{So } \nu(q_1 \circ l \circ p_1 - q_1 \circ f_1) = 0 \text{ and it}$$

factors through the kernel of  $\nu q_1$ ,  $\tau N$  as  $m$ .

$$\text{Now } \text{ker } \nu q_1 \circ m \circ \text{ker } \nu p_1 = \nu f_1 \circ \text{ker } \nu p_1 \text{ ( + } \nu l \circ \nu p_1 \circ \text{ker } \nu p_1 \text{ )} = \text{ker } \nu q_1 \circ \tau f.$$

$\Rightarrow \tau f = m \circ \text{ker } \nu p_1$  and  $\tau f$  factors through as injective.  $\square$

### Corollary

$\tau$  is well-defined on objects (if we take minimal presentation)

**Pf.**

$\tau M$  does not have any non-zero injective summands. Objects in  $\text{mod } A$  without injective summands are determined by their image in  $\overline{\text{mod } A}$ .  $\square$

### Corollary

$M$  indec non-proj, then  $\text{End}(M) / \text{rad End}(M)$  is a skew-field, and isomorphic to  $\text{End}(\tau M) / \text{rad End}(\tau M)$ .

*Pf.*  
 From equivalence by  $\gamma$ .  
 $\text{End}(M)/\text{rad End}(M) \cong \overline{\text{End}(\tau M)}/\text{rad } \overline{\text{End}(\tau M)}$   
 $\cong \text{End}(\tau M)/\text{rad End}(\tau M)$   
 All maps which factors through a projective is in the radical.  $\square$   
Warning: In general,  $\text{End}(M)$  and  $\text{End}(\tau M)$  might be different.

follows from indecomposable

Austlander's defect formula:

Let  $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$  be a short exact seq. and  $A \in \text{mod } A$ .

\* The covariant defect is:

$$\text{cov def} = \text{Coh}([\text{Hom}(X, M) \rightarrow \text{Hom}(X, N)])$$

\* The contravariant defect is:

$$\text{contra def} = \text{Coh}([\text{Hom}(M, X) \rightarrow \text{Hom}(L, X)])$$

→ These are functors in both  $X$  and the short exact seq.

Remark

In  $\text{Fun}(\text{mod } A^{\text{op}}, \text{mod } K)$  we have a proj. res. of cov def:

$$0 \rightarrow \text{Hom}(-, L) \rightarrow \text{Hom}(-, M) \rightarrow \text{Hom}(-, N)$$

Thm (Austlander's defect formula)

$0 \text{ cov def}(X) \cong \text{contra def}(\tau X)$ , is natural in both arguments.

*Pf.*

Pick a proj. presentation of  $X$ :  $P_1 \rightarrow P_0 \rightarrow X \rightarrow 0$

$$\begin{array}{ccccccc} 0 & \rightarrow & \text{Hom}(X, L) & \rightarrow & \text{Hom}(X, M) & \rightarrow & \text{Hom}(X, N) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & \text{Hom}(P_0, L) & \rightarrow & \text{Hom}(P_0, M) & \rightarrow & \text{Hom}(P_0, N) \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & \text{Hom}(P_1, L) & \rightarrow & \text{Hom}(P_1, M) & \rightarrow & \text{Hom}(P_1, N) \rightarrow 0 \end{array}$$

We dualize the diagram:

$$\begin{array}{ccccccc}
 & & 0 & & & & \\
 & & \uparrow & & & & \\
 0 & \leftarrow & D\text{Hom}(X, L) & \leftarrow & D\text{Hom}(X, M) & \leftarrow & D\text{Hom}(X, N) & \leftarrow & D\text{condef}(X) & \leftarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & & & \\
 0 & \leftarrow & D\text{Hom}(P_0, L) & \leftarrow & D\text{Hom}(P_0, M) & \leftarrow & D\text{Hom}(P_0, N) & \leftarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 0 & \leftarrow & \text{Hom}(L, \nu P_0) & \leftarrow & \text{Hom}(M, \nu P_0) & \leftarrow & \text{Hom}(N, \nu P_0) & \leftarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 0 & \leftarrow & D\text{Hom}(P_i, L) & \leftarrow & D\text{Hom}(P_i, M) & \leftarrow & D\text{Hom}(P_i, N) & \leftarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 0 & \leftarrow & \text{Hom}(L, \nu P_i) & \leftarrow & \text{Hom}(M, \nu P_i) & \leftarrow & \text{Hom}(N, \nu P_i) & \leftarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 \text{condef}(\tau X) & \leftarrow & \text{Hom}(L, \tau X) & \leftarrow & \text{Hom}(M, \tau X) & \leftarrow & \text{Hom}(N, \tau X) & \leftarrow & 0
 \end{array}$$

By gluing the black and orange diagram together, we get the isomorphism by snake lemma.  $\square$

Exercise

\* Find  $\tau$  of the indec. of the following quivers:

$$\left\{ \begin{array}{c} \bullet \\ \swarrow \downarrow \searrow \\ \bullet \end{array} \right\} \quad \& \quad \left\{ \begin{array}{c} \bullet \\ \swarrow \downarrow \nearrow \\ \bullet \end{array} \right\}$$

\* If  $A$  is a symmetric algebra, i.e.  $A \cong DA$ , then  $\tau \cong \Omega$ .