Repteori - Week 4.

$\nu^{-} = \mathrm{Hom}_A(D-, A)$ is the opposite order of ν.

Notice that $\nu^{-} \simeq \mathrm{Hom}_A(DA, -)$, and is right adjoint to $\nu : \mathrm{mod}_A \to \mathrm{mod}_A$ s.t. ν restricts to an equivalence $\nu|_{\mathrm{proj} A} : \mathrm{proj}\, A \to \mathrm{inj}\, A$.

Construction:

$M \in \mathrm{mod}\, A$, Pick a minimal projective presentation:

$P_1 \xrightarrow{p} P_0 \to M \to 0$. Define τ as $\ker \nu p$, i.e.

$$0 \to \tau M \to \nu P_1 \xrightarrow{\nu p} \nu P_0.$$

Remark: τ is also known as DTr — Tr is called the trace

$Tr\,M$ is defined as the cokernel of $\mathrm{Hom}(p, A)$.

Another remark: We can do the inverse construction to get τ^{-} from ν^{-1}. Pick an injective copresentation, and define τ^{-1} as $\mathrm{cok}\, \nu^{-1} i$:

$$0 \to M \to I^0 \xrightarrow{i} I^1$$

$$\downarrow$$

$$\nu^{-1} I^0 \xrightarrow{\nu^{-1} i} \nu^{-} I^1 \to \tau^{-} M \to 0.$$

Ex. $A = \mathbb{K}[1 \to 2]$, $M = [0 \leftarrow \mathbb{K}]$. Construct τ of M.

$$\{\mathbb{K} \leftarrow 0\} \xrightarrow{*^0} \{\mathbb{K} \leftarrow \mathbb{K}\} \to M \to 0$$

$$\downarrow \nu$$

$$0 \to \{\mathbb{K} \leftarrow 0\} \to \{\mathbb{K} \leftarrow \mathbb{K}\} \to \{0 \leftarrow \mathbb{K}\}$$

- A f.d. and P proj.

$$0 \to P \to P \to 0$$

$$\downarrow$$

$$0 \to 0 \to 0 \to \nu P \quad \Rightarrow \quad \tau P = 0.$$

Def. *Projective stable*

The stable module category $\underline{\mathrm{mod}}^A$ is given by:

- $\to \mathrm{Obj}(\underline{\mathrm{mod}}^A) = \mathrm{Obj}(\mathrm{mod}^A)$
- $\to \underline{\mathrm{Hom}}_A(M,N) = \mathrm{Hom}_A(M,N) /$ maps factoring through a projective.

If P is projective, then $P \simeq 0$ in $\underline{\mathrm{mod}}^A$.

Dually, $\overline{\mathrm{mod}}^A$ is given by: *Injective stable (costable ?)*

- $\to \mathrm{Obj}(\overline{\mathrm{mod}}^A) = \mathrm{Obj}(\mathrm{mod}^A)$
- $\to \overline{\mathrm{Hom}}_A(M,N) = \mathrm{Hom}_A(M,N) /$ maps factoring throug an injective

Thm *

τ and τ^- induce mutually inverse equivalences:

$$\begin{tikzcd} \underline{\mathrm{mod}}^A \arrow[r, bend right, "\tau"'] & \overline{\mathrm{mod}}^A \arrow[l, bend right, "\tau^-"'] \end{tikzcd}$$

~ Start of lecture ~

Calculating projectives and injectives:

* $\mathbb{K}\left[\begin{tikzcd} 1 \arrow[r, "a"] \arrow[d, "c"'] & 2 \arrow[d, "b"] \\ 3 \arrow[r, "d"'] & 4 \end{tikzcd}\right] / \langle ba - dc \rangle$

Simples:
$\begin{tikzcd} \mathbb{K} & 0 \arrow[l] \\ 0 \arrow[u] & 0 \arrow[l] \arrow[u] \end{tikzcd}$,
$\begin{tikzcd} 0 & \mathbb{K} \arrow[l] \\ 0 \arrow[u] & 0 \arrow[l] \arrow[u] \end{tikzcd}$,
$\begin{tikzcd} 0 & 0 \arrow[l] \\ \mathbb{K} \arrow[u] & 0 \arrow[l] \arrow[u] \end{tikzcd}$,
$\begin{tikzcd} 0 & 0 \arrow[l] \\ 0 \arrow[u] & \mathbb{K} \arrow[l] \arrow[u] \end{tikzcd}$

indec. Proj:
$\begin{tikzcd} \mathbb{K} & 0 \arrow[l] \\ 0 \arrow[u] & 0 \arrow[l] \arrow[u] \end{tikzcd}$,
$\begin{tikzcd} \mathbb{K} & \mathbb{K} \arrow[l] \\ 0 \arrow[u] & 0 \arrow[l] \arrow[u] \end{tikzcd}$,
$\begin{tikzcd} \mathbb{K} & 0 \arrow[l] \\ \mathbb{K} \arrow[u] & 0 \arrow[l] \arrow[u] \end{tikzcd}$,
$\begin{tikzcd} \mathbb{K} & \mathbb{K} \arrow[l] \\ \mathbb{K} \arrow[u] & \mathbb{K} \arrow[l] \arrow[u] \end{tikzcd}$

indec. inj:
$\begin{tikzcd} \mathbb{K} & \mathbb{K} \arrow[l] \\ \mathbb{K} \arrow[u] & \mathbb{K} \arrow[l] \arrow[u] \end{tikzcd}$,
$\begin{tikzcd} 0 & \mathbb{K} \arrow[l] \\ 0 \arrow[u] & \mathbb{K} \arrow[l] \arrow[u] \end{tikzcd}$,
$\begin{tikzcd} 0 & 0 \arrow[l] \\ \mathbb{K} \arrow[u] & \mathbb{K} \arrow[l] \arrow[u] \end{tikzcd}$.
$\begin{tikzcd} 0 & 0 \arrow[l] \\ 0 \arrow[u] & \mathbb{K} \arrow[l] \arrow[u] \end{tikzcd}$

Continuing from thm *

τ and τ^- are quasi-inverses if well-defined, as they undo each other by construction. Well-definedness is however the only obstacle.

Pick a morphism $f: M \to N$.

$$\begin{tikzcd} P_1 \arrow[r] \arrow[d, dotted, "\exists f_1"] & P_0 \arrow[r, two heads] \arrow[d, dotted, "\exists f_0"] & M \arrow[r] \arrow[d, "f"] & 0 \\ Q_1 \arrow[r] & Q_0 \arrow[r, two heads] & N \arrow[r] & 0 \end{tikzcd}$$

Maps between $P_i \to Q_i$ obtained from projective resolution property. See homalg notes for more info.

Apply ν to diagram:

$$\begin{tikzcd}
0 \arrow[r] & \tau M \arrow[r, tail] \arrow[d, dashed, "\tau f"] & \nu P_1 \arrow[r] \arrow[d, "\nu f_1"] & \nu P_0 \arrow[d, "\nu f_0"] \\
0 \arrow[r] & \tau N \arrow[r, tail] & \nu Q_1 \arrow[r] & \nu Q_0
\end{tikzcd}$$

kernel prop

Problem: f_0, f_1 not unique, so νf_i not unique either. τf is called our candidate.

Subtracting different choices for the same f. Can assume $f = 0$, need to show $\tau f = 0$, indep. choices.

$f = 0$ in $\underline{\text{mod}}^A \Leftrightarrow f$ factors through $Q_0 \to N$ by h.

f factors through projective.

$f_0 = h \circ p_0 + q_1 \circ l$

$$\begin{tikzcd}
P_1 \arrow[r, "p_1"] \arrow[d, "f_1"] & P_0 \arrow[r, two heads, "p_0"] \arrow[d, "f_0"] \arrow[dl, "l"'] & M \arrow[d, "f"] \arrow[dl, "h"'] \\
Q_1 \arrow[r, "q_1"'] & Q_0 \arrow[r, two heads, "q_0"'] & N
\end{tikzcd}$$

$q_0 \circ f_0 = f \circ p_0 = q_0 \circ h \circ p_0$

$\Rightarrow f_0 - h \circ p_0$ factors through $\operatorname{Ker} q_0$.

As P_0 proj: even factors through proj cover of kernel, i.e. through q_1 by l.

$\downarrow \nu$

$$\begin{tikzcd}
\tau M \arrow[r, tail] \arrow[d, dashed, "\tau f"] & \nu P_1 \arrow[r, "\nu p_1"] \arrow[d, "\nu f_1"] \arrow[dl, "m"'] & \nu P_0 \arrow[d, "\nu f_0"] \arrow[dl, "\nu l"'] \\
\tau N \arrow[r, tail] & \nu Q_1 \arrow[r, "\nu q_1"'] & \nu Q_0
\end{tikzcd}$$

In particular, $f_0 \circ p_1 = q_1 \circ l \circ p_1$

$\Rightarrow \nu f_0 \circ \nu p_1 = \nu q_1 \circ \nu l \circ \nu p_1 = \nu q_1 \circ \nu f_1$

So $\nu(q_1 \circ l \circ p_1 - q_1 \circ f_1) = 0$ and it factors through the kernel of νq_1, τN as m.

Now $\ker \nu q_1 \circ m \circ \ker \nu p_1 = \nu f_1 \circ \ker \nu p_1 \; (+\, \nu l \circ \overbrace{\nu p_1 \circ \ker \nu p_1}^{0}) = \ker \nu q_1 \circ \tau f$.

$\Rightarrow \tau f = m \circ \ker \nu p_1$ and τf factors through an injective. $\square$

Corollary

τ is well-defined on objects (if we take minimal presentation)

Pf.

τM does not have any non-zero injective summands. Objects in mod^A without injective summands are determined by their image in $\overline{\text{mod}}^A$. $\square$

Corollary

M indec non-proj, then $\operatorname{End}(M)/\operatorname{rad}\operatorname{End}(M)$ is a skew-field, and isomorphic to $\operatorname{End}(\tau M)/\operatorname{rad}\operatorname{End}(\tau M)$.

Pf.

From equivalence by τ.

$$\mathrm{End}(M)/\mathrm{rad}\,\underline{\mathrm{End}}(M) \simeq \overline{\mathrm{End}}(\tau M)/\mathrm{rad}\,\overline{\mathrm{End}}(\tau M)$$
$$\mathrm{End}(M)/\mathrm{rad}\,\mathrm{End}(M) \qquad \mathrm{End}(\tau M)/\mathrm{rad}\,\mathrm{End}(\tau M)$$

follows from indecomposable → All maps which factors through a projective is in the radical.

□

Warning: In general, End(M) and End(τM) might be different.

Auslander's defect formula:

Let $0 \to L \rightarrowtail M \twoheadrightarrow N \to 0$ be a short exact seq. and $X \in \mathrm{mod}\,A$.

* The covariant defect is:

$$\mathrm{cov\,def} = \mathrm{Coh}([\mathrm{Hom}(X,M) \to \mathrm{Hom}(X,N)])$$

* The contravariant defect is:

$$\mathrm{contra\,def} = \mathrm{Coh}([\mathrm{Hom}(M,X) \to \mathrm{Hom}(L,X)])$$

→ These are functors in both X and the short exact seq.

Remark
In $\mathrm{Fun}((\mathrm{mod}\,A)^{op}, \mathrm{mod}\,\mathbb{K})$ we have a proj. res. of cov def:

$$0 \to \mathrm{Hom}(-,L) \to \mathrm{Hom}(-,M) \to \mathrm{Hom}(-,N).$$

Thm (Auslander's defect formula)

$D\,\mathrm{cov\,def}(X) \cong \mathrm{contra\,def}(\tau X)$, is natural in both arguments.

Pf.
Pick a proj. presentation of X: $P_1 \to P_0 \twoheadrightarrow X \to 0$

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}(X,L) \arrow[r, tail] \arrow[d, tail] & \mathrm{Hom}(X,M) \arrow[r] \arrow[d, tail] & \mathrm{Hom}(X,N) \arrow[d, tail] & \\
0 \arrow[r] & \mathrm{Hom}(P_0,L) \arrow[r, tail] \arrow[d] & \mathrm{Hom}(P_0,M) \arrow[r, two heads] \arrow[d] & \mathrm{Hom}(P_0,N) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Hom}(P_1,L) \arrow[r, tail] & \mathrm{Hom}(P_1,M) \arrow[r, two heads] & \mathrm{Hom}(P_1,N) \arrow[r] & 0
\end{tikzcd}$$

We dualize the diagram:

$$
\begin{tikzcd}
 & 0 & & & & \\
0 & D\,\mathrm{Hom}(X,L) \arrow[l] \arrow[u] & D\,\mathrm{Hom}(X,M) \arrow[l] & D\,\mathrm{Hom}(X,N) \arrow[l] & D\,\mathrm{cor\,def}(X) \arrow[l] & 0 \arrow[l] \\
0 & D\,\mathrm{Hom}(P_0,L) \arrow[l] \arrow[u] & D\,\mathrm{Hom}(P_0,M) \arrow[l] \arrow[u] & D\,\mathrm{Hom}(P_0,N) \arrow[l] \arrow[u] & 0 \arrow[l] & \\
0 & \mathrm{Hom}(L,\nu P_0) \arrow[l] \arrow[u, equal] & \mathrm{Hom}(M,\nu P_0) \arrow[l] \arrow[u, equal] & \mathrm{Hom}(N,\nu P_0) \arrow[l] \arrow[u, equal] & & \\
0 & D\,\mathrm{Hom}(P_1,L) \arrow[l] \arrow[uu] & D\,\mathrm{Hom}(P_1,M) \arrow[l] \arrow[uu] & D\,\mathrm{Hom}(P_1,N) \arrow[l] \arrow[uu] & 0 \arrow[l] & \\
0 & \mathrm{Hom}(L,\nu P_1) \arrow[l] \arrow[u, equal] \arrow[uu] & \mathrm{Hom}(M,\nu P_1) \arrow[l] \arrow[u, equal] \arrow[uu] & \mathrm{Hom}(N,\nu P_1) \arrow[l] \arrow[u, equal] \arrow[uu] & & \\
\mathrm{contradef}(\tau X) & \mathrm{Hom}(L,\tau X) \arrow[l] \arrow[u] & \mathrm{Hom}(M,\tau X) \arrow[l] \arrow[u] & \mathrm{Hom}(N,\tau X) \arrow[l] \arrow[u] & &
\end{tikzcd}
$$

By gluing the black and orange diagram together, we get the isomorphism by snake lemma. □

Exercises

* Find τ of the indec. of the following quivers:

$$
\left\{\begin{tikzcd}[sep=small] & \bullet \arrow[dl] \arrow[dr] & \\ \bullet & & \bullet \end{tikzcd}\right\}
\quad \& \quad
\left\{\begin{tikzcd}[sep=small] & \bullet \arrow[dl] \arrow[d] \arrow[dr] & \\ \bullet & \bullet & \bullet \end{tikzcd}\right\}
$$

* If A is a symmetric algebra, i.e. $A \simeq DA$, then $\tau \simeq \Omega^2$.