Repteori - Week 5

Exercises:

* $Q = \left\{ \begin{smallmatrix} & 0 & \\ \swarrow & & \searrow \\ 1 & & 2 \end{smallmatrix} \right\}$ We know the indec. τ of proj is 0. ok

$\tau\left(\begin{smallmatrix} & 0 & \\ \nearrow & & \nwarrow \\ \mathbb{K} & & 0 \end{smallmatrix} \right) = \underline{? = \tau(i_1)}$

Projectives

Let $P_0 = \begin{smallmatrix} & \mathbb{K} & \\ \nearrow & & \nwarrow \\ 0 & & 0 \end{smallmatrix}$, $P_1 = \begin{smallmatrix} & \mathbb{K} & \\ \nearrow & & \nwarrow \\ \mathbb{K} & & 0 \end{smallmatrix}$, $P_2 = \begin{smallmatrix} & \mathbb{K} & \\ \nearrow & & \nwarrow \\ 0 & & \mathbb{K} \end{smallmatrix}$

$i_1 = \begin{smallmatrix} & 0 & \\ \nearrow & & \nwarrow \\ \mathbb{K} & & 0 \end{smallmatrix}$, $i_2 = \begin{smallmatrix} & 0 & \\ \nearrow & & \nwarrow \\ 0 & & \mathbb{K} \end{smallmatrix}$, $i_0 = \begin{smallmatrix} & \mathbb{K} & \\ \nearrow & & \nwarrow \\ \mathbb{K} & & \mathbb{K} \end{smallmatrix}$ Injectives

* Pick projective presentation for i_1:

$$P_0 \to P_1 \twoheadrightarrow i_1$$

* Apply ν and find kernel:

$$P_2 \hookrightarrow i_0 \to i_1$$

Thus $\tau(i_1) = P_2$.

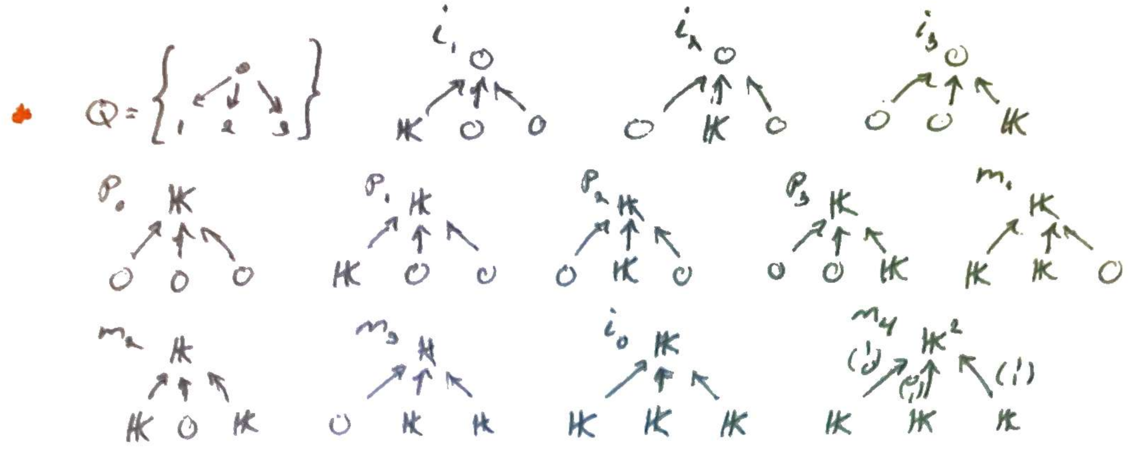
$\rightsquigarrow \underline{\tau(i_2) = P_1}$ Symmetry from i_1.

$\underline{\tau(i_0) = ?}$

* Pick projective pres: $P_0 \to P_1 \oplus P_2 \twoheadrightarrow i_0 \to 0$

* Apply ν: $P_0 \hookrightarrow i_0 \to i_1 \oplus i_2$

Thus $\tau(i_0) = P_0$.

* 

Again τ of proj are 0. ok

$i_1 \rightsquigarrow \tau(i_1) = m_3$

* pres : $p_0 \hookrightarrow p_1 \twoheadrightarrow i_1$ * ν : $m_3 \hookrightarrow i_0 \to i_1$

$\Rightarrow$ $i_2 \rightsquigarrow \tau(i_2) = m_2$, $i_3 \rightsquigarrow \tau(i_3) = m_1$

$m_1 \rightsquigarrow \tau(m_1) = p_3$

* pres : $p_0 \hookrightarrow p_1 \oplus p_2 \twoheadrightarrow m_1$ * ν : $p_3 \hookrightarrow i_0 \to i_1 \oplus i_2$

$m_2 \rightsquigarrow \tau(m_2) = p_2$, $m_3 \rightsquigarrow \tau(m_3) = p_1$

$i_0 \rightsquigarrow \tau(i_0) = m_4$

* pres : $p_0^2 \hookrightarrow p_1 \oplus p_2 \oplus p_3 \twoheadrightarrow i_0$ * ν : $m_4 \hookrightarrow i_0^2 \to i_1 \oplus i_2 \oplus i_3$

$m_4 \rightsquigarrow \tau(m_4) = p_0$

* pres : $p_0 \hookrightarrow p_1 \oplus p_2 \oplus p_3 \twoheadrightarrow m_4$ * ν : $p_0 \hookrightarrow i_0 \to i_1 \oplus i_2 \oplus i_3$

* Assume $A \simeq DA$ as A-A-bimodules and A f.d.
WTS : $\tau \simeq \Omega^2$.

We calculate $\tau(M)$ as applying ν to proj pres, but $\nu \simeq \mathrm{Id}$, so τ is then taking kernel of a projective pres, which is taking syzygy of kernel, which is the second syzygy.

$$\begin{tikzcd} \Omega^2 M \arrow[r, hook] & P_1 \arrow[rr] \arrow[dr, two heads] & & P_0 \arrow[r, two heads] & M \\ & & \Omega M \arrow[ur, hook] & & \end{tikzcd}$$

Defect formula

- $D(\text{cov defect}(X)) \simeq \text{contrav defect}(\tau X)$

We want to apply this to:

Finding connections to Ext.

$$0 \to \Omega M \xrightarrow{b} P \xrightarrow{p} M \to 0$$

$$\text{cov def}(X) = \text{Cok}\,\text{Hom}(X, p) \simeq \underline{\text{Hom}}(X, M)$$

$$\text{contra def}(Y) = \text{Cok}\,\text{Hom}(b, Y) \simeq \text{Ext}^1(M, Y)$$

Note: There is a long exact sequence

$$0 \to \text{Hom}(M, Y) \to \text{Hom}(P, Y) \to \text{Hom}(\Omega M, Y) \to \text{Ext}^1(M, Y) \to \overset{0}{\cancel{\text{Ext}^1(P, Y)}} \to \dots$$

Corollary (Auslander-Reiten formula)

$$D\,\underline{\text{Hom}}_A(M, N) \simeq \text{Ext}^1_A(N, \tau M)$$

$$D\,\overline{\text{Hom}}_A(M, N) \simeq \text{Ext}^1_A(\tau N, M)$$

This follows from the above discussion

Almost split sequences (Auslander-Reiten sequences)

Def. $f: M \to N$ is right almost split if:

- is not split-epi
- any other map $g: M' \to N$ that is not split-epi, factors through f.

This is not a universal property as we dont have uniqueness.

$f: M \to N$ is left almost split if:

- is not split-mono
- any other map $g: M \to N'$ that is not split-mono, factors through f.

Def. A short exact sequence is almost split if

- a is left almost split
- b is right almost split

$$0 \to L \xrightarrow{a} M \xrightarrow{b} N \to 0$$

Obs

- $f: M \to N$ right almost split $\Rightarrow$ N indec.
- f left almost split $\Rightarrow$ M indec.

If $N \simeq N_1 \oplus N_2$, $N_i \neq 0$, then

$$\begin{tikzcd} M \arrow[r, "f"] & N & \\ & N_1 \arrow[ul, dashed] \arrow[u, bend right, "h_1"'] & N_2 \arrow[ul, bend right, "h_2"'] \arrow[ull, dashed] \end{tikzcd}$$

$h_1 \oplus h_2$ give a factorization of N
$\Rightarrow$ f is split-epi ↯

Prop
If $0 \to L \rightarrowtail M \twoheadrightarrow N \to 0$ is almost split, then $L \simeq \tau N$.

Pf
$\text{cordef}(X) \simeq 0$, but $\text{cordef}(N) \neq 0$.
Assume X indec. $X \not\simeq N$

Dually: $\text{contradef}(X) \simeq 0$, but $\text{contradef}(L) \neq 0$.
Assume X indec. $X \not\simeq L$

By the defect formula: $\text{cov def}(X) \neq 0 \Leftrightarrow \text{contradef}(\tau X) \neq 0$.

E.g. $\text{covdef}(N) \neq 0 \Leftrightarrow \text{contradef}(\tau N) \neq 0$, so in particular $\tau N \simeq L$. $\square$

Prop
$0 \to \tau N \overset{a}{\rightarrowtail} M \overset{b}{\twoheadrightarrow} N$ is a short exact sequence.
The following are equivalent:

- a is left almost split
- b is right almost split
- The sequence is almost split

Pf

Assume b is right almost split.
$\Leftrightarrow$ $\text{cordef}(X) \simeq 0$ if $X \not\simeq N$ or $\text{End}(N)/\text{rad}\,\text{End}(N)$ if $X \simeq N$ for indec. X.

a is left almost split.
$\Leftrightarrow$ $\text{contradef}(X) \simeq 0$ if $X \not\simeq \tau N$ or $\text{End}(\tau N)/\text{rad}\,\text{End}(\tau N)$ if $X \simeq \tau N$ for indec. X.

By the defect formula and the fact that $\dim \text{End}(N)/\text{rad}\,\text{End}(N) = \dim \text{End}(\tau N)/\text{rad}\,\text{End}(\tau N)$, these statements are equivalent. $\square$

Ex If P indec. proj. then $\operatorname{rad} P \hookrightarrow P$ is right almost split.
The radical is the biggest proper submodule of P,
so any non-epi factors throug $\operatorname{rad} P$.

Dually I indec. inj. $I \twoheadrightarrow I/\operatorname{rad} I$ is left almost split.

End of lecture $\sim$ Start of lecture

Thm.
Almost split sequences always exist.
i.e. for any non-projective indec. N, there is an almost split sequence:

$$0 \to \tau N \rightarrowtail M \twoheadrightarrow N \to 0$$

Pf. natural — AR-formula

$$D\,\underline{\operatorname{Hom}}(N,N) \simeq \operatorname{Ext}^1(N,\tau N)$$

Pick $\varphi \in D\,\underline{\operatorname{Hom}}(N,N)$ s.t. $\varphi \neq 0$ and $\varphi|_{\operatorname{rad}\underline{\operatorname{End}}(N)} = 0$

By AR iso $\varphi \mapsto E$ — Remember that E is a short exact seq.

Claim E is almost split — Enough to show that E is non-split and right almost split.

$\varphi \neq 0 \iff E \neq 0$ i.e. E is non-split

Let $f : T \to N$ be non split-epi.

Want:

$$\begin{tikzcd} E: & \tau N \arrow[r, tail] & M \arrow[r, two heads] & N \\ & & \bullet \arrow[u] \arrow[r] \arrow[ur, phantom, "\lrcorner"] & T \arrow[u] \arrow[ul, dashed, "\exists"'] \end{tikzcd}$$

E is almost split

$\iff$ Pullback along E of f splits

$\iff$ $E \cdot f = 0$

Use natural iso from AR-formula

$$\begin{tikzcd} \varphi \arrow[d, mapsto] & D\,\underline{\operatorname{Hom}}(N,N) \arrow[r, phantom, "\simeq"] \arrow[d, dashed] & \operatorname{Ext}^1(N,\tau N) \arrow[d, "f^*"] & E \arrow[d, mapsto] \\ {[g \mapsto \varphi(fg)]} & D\,\underline{\operatorname{Hom}}(N,T) \arrow[r, phantom, "\simeq"] & \operatorname{Ext}^1(T,\tau N) & E \cdot f \end{tikzcd}$$

$\varphi \longmapsto E$, $[g \to \varphi(fg)] \longmapsto E \cdot f$

Pick $\varphi \in D\,\underline{\operatorname{Hom}}(N,N)$, then $[g \mapsto \varphi(fg)]$

Claim $[g \to \varphi(fg)]$ if φ is picked as $\varphi \neq 0$ and $\varphi|_{\operatorname{rad}\underline{\operatorname{End}}(N)} = 0$.

f is not split-epi, but fg is. $\underline{\operatorname{End}}(N)$ is local
$\Rightarrow$ (fg) is in $\operatorname{rad}\underline{\operatorname{End}}(N)$

$\Rightarrow \varphi(fg) = 0$ for any g. $\square$

Obs

The almost split sequence ending in N is unique up to isomorphism.

Suppose there are two choices:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, equal] & \tau N \arrow[r] \arrow[d, dashed, shift right, "d"'] & M \arrow[r] \arrow[d, dashed, shift right, "b"'] & N \arrow[r] \arrow[d, equal] & 0 \arrow[d, equal] \\
0 \arrow[r] & \tau N \arrow[r] \arrow[u, dashed, shift right, "c"'] & M' \arrow[r] \arrow[u, dashed, shift right, "a"'] & N \arrow[r] & 0
\end{tikzcd}$$

We combine morphisms to get new maps.

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, equal] & \tau N \arrow[r] \arrow[d, "cd"] & M \arrow[r] \arrow[d, "ab"] & N \arrow[r] \arrow[d, equal] & 0 \arrow[d, equal] \\
0 \arrow[r] & \tau N \arrow[r] & M \arrow[r] & N \arrow[r] & 0
\end{tikzcd}$$

cd is either iso or nilpotent, as $\operatorname{End}(\tau N)$ is local.

Assume that cd is nilpotent, then $(cd)^n = 0$, and the lower sequence is therefore split, as cd, ab square is a pushout. This contradicts the fact that the sequence is almost split.

Thus cd is iso, and ab is iso by 5-lemma, so c is in particular iso as well. $\square$

Def.

A map $f: M \to N$ is right minimal if $\forall \varphi \in \operatorname{End}(M)$ s.t. $f = f\varphi$, then φ is an iso.

A map $f: M \to N$ is left minimal if $\forall \varphi \in \operatorname{End}(N)$ s.t. $f = \varphi f$, then φ is an iso.

Exercise

If $f: M \to N$, then there is a decomposition $M = M_1 \oplus M_2$, s.t. $f = (\underset{\text{right minimal}}{f_1} \quad 0)$ ← show this.

Obs

$0 \to \tau N \xrightarrow{a} M \xrightarrow{b} N \to 0$ almost split, then a is left minimal, and b is right minimal.

Thm $0 \to L \to M \to N \to$ is a short exact seq.

TFAE:

* The sequence is almost split
* b is right almost split & $L \simeq \tau N$
* b is right almost split & L is indec.
* b is right minimal almost split

~ duals

* a is left almost split & $N \simeq \tau^{-}L$
* a is left almost split & N is indec
* a is left minimal almost split

Pf. We show that everything are equivalent.

Almost split seq $\Leftrightarrow$ b right almost split, $L \simeq \tau N$ ✓

We miss implications to almost split:

b right almost split, L indec $\Rightarrow$ $L \to M \to N$ almost split.

We have the seq

This exists

$$\begin{tikzcd} L \arrow[r, "a"] \arrow[d, shift right, "f"'] & M \arrow[r, "b"] \arrow[d, shift right] & N \arrow[d, equal] \\ \tau N \arrow[r] \arrow[u, shift right, "g"'] & E \arrow[r] \arrow[u, shift right] & N \end{tikzcd}$$

These exists by right almost splitting

As L is indec. gf is either iso or nilpotent, but we have seen that gf cannot be iso. So the sequences are isomorphic.

b is right minimal almost split $\Rightarrow$ $L \to M \to N$ almost split.

Do the same:

$$\begin{tikzcd} L \arrow[r, "a"] \arrow[d, shift right] & M \arrow[r, "b"] \arrow[d, shift right, "f"'] & N \arrow[d, equal] \\ \tau N \arrow[r] \arrow[u, shift right] & E \arrow[r] \arrow[u, shift right, "g"'] & N \end{tikzcd}$$

As b is minimal, gf is iso, but $E \to N$ is also minimal, so fg is iso. Thus the kernels are iso, and the sequences are iso. □

Irreducible morphisms

Def. $f: M \to N$ is irreducible if f is neither split-mono or split-epi, and whenever $f = hg$, either h is split-epi or g is split-mono.

Obs

If f is irreducible, then f is mono or epi.

Prop

N is indec. $f: M \to N$, then f is irreducible if and only if $\exists f': M' \to N$ s.t. $(f\ f'): M \oplus M' \to N$ is right minimal almost split.

Lemma

For any indec. N $\exists$ right minimal almost split map $E \to N$!

Pf

N proj $\rightsquigarrow$ $\operatorname{rad} N \to N$

N otherwise $\rightsquigarrow$ $\tau N \to E \to N$ almost split $\square$

Pf "$\Rightarrow$" Assume f irred.

$f = ?g$, so g is split mono.

$$\begin{tikzcd} & M \arrow[dl, dashed, "g"'] \arrow[d, "f"] \\ E \arrow[r, "\substack{\text{right}\\\text{minimal}\\\text{almost split}}"'] & N \end{tikzcd}$$

"$\Leftarrow$" Assume $(f\ f'): M \oplus M' \to N$ is right minimal almost split.

$$\begin{tikzcd} M \arrow[rr, "f"] \arrow[dr, "g"'] & & N \\ & X \arrow[ur, "h"'] & \end{tikzcd}$$

Assume factorization exists w/ h not split-epi.

$$\begin{tikzcd} M \oplus M' \arrow[rr, "(f\ f')"] \arrow[dr, "g \oplus \mathrm{id}"'] & & N \\ & X \oplus M' \arrow[ul, dashed, "l"] \arrow[ur, "(h\ f')"'] & \end{tikzcd}$$

$(h\ f')$ is not split epi.

We now have that $(f\ f')\, l\, (g \oplus \mathrm{id}) = (f\ f')$

$\Rightarrow l(g \oplus \mathrm{id})$ is iso, so $(g \oplus \mathrm{id})$ is split-mono. $\square$

Corollary

M indec. and non-inj, N indec and non-proj. Then M appears as the direct summand of the middle term of the the almost split sequence ending in N if and only if N appears as a direct summand of the middle term in the almost split sequence starting in M.

Pf Both statements are equivalent that there is an irreducible map from $M \to N$.

i.e. pick $f: M \to N$ irred $\Leftrightarrow$ $\exists f'$ s.t. $(f\ f'): M \oplus M' \to N$ is the final morphism in the almost split sequence ending in N. $\square$

A f.d. algebra

Def. The Auslander-Reiten quiver Γ_A of A is given by:

vertices = indecomposables

arrows = existence of irreducible maps

dashed arrows = action of τ.

An arrow $M \xrightarrow{(a,b)} N$ gets labeled (a,b), where a there are copies of M appears in almost split seq. ending in N, and b copies of N appears in almost split seq. starting in M.

We drop (a,b) if $a=b=1$, and write n arrows if $a=b=n$.

Exercise
If M indec. proj & inj, then $\exists$ seq as

$$0 \to \operatorname{rad} M \to M \oplus \operatorname{rad} M / \operatorname{soc} M \to M / \operatorname{soc} M \to 0$$