

Reptori - Week 6

Exercise 1

Let $f: M \rightarrow N$ be f.g. modules over a f.d. algebra

$\text{SI } \begin{matrix} \nearrow f|_{M_1}, f|_{M_2} \\ M_1 \oplus M_2 \end{matrix}$ where $f|_{M_1} = 0$.

want to show this

If M has direct summand that is 0 under f , it is a part of M_2 . May assume that $f|M_i \neq 0$ for any summand M_i .

$f\varphi$ is right minimal if $\forall \psi \in \text{End } M$ s.t. $f = f\varphi\psi$, then $f\varphi$ is an iso. Moreover $f \circ \varphi^n = f \quad \forall n \in \mathbb{N}$. Now,

$\ker \varphi \subseteq \ker \varphi^2 \subseteq \dots \subseteq \ker \varphi^n = \ker \varphi^{n+1} = \dots$

Fitting Lemma

$\varphi: M^{\oplus n}$, M has finite length, for $n \geq 0$

$$M \cong \ker \varphi^n \oplus \text{Im } \varphi^n$$

With this lemma, $f = f\varphi^n$, so $f|_{\ker \varphi^n} = 0$
 $\Rightarrow \ker \varphi^n \cong 0$ by assumption. By f.d. φ is an iso.

Pf.

Pick n s.t. $\ker \varphi^n = \ker \varphi^m$ for $m > n$
 $\text{Im } \varphi^n = \text{Im } \varphi^m$ for $m > n$.

$$\begin{array}{ccccc}
 M & \xrightarrow{\varphi^n} & M & \xrightarrow{\varphi^n} & M \\
 \downarrow & \nearrow \text{Im } \varphi^n & \downarrow & \nearrow \text{Im } \varphi^n & \downarrow \\
 & & \text{Im } \varphi^n & & \text{Im } \varphi^n \\
 & & \cong \text{Im } \varphi^n & & \text{Im } \varphi^n \\
 & & \xrightarrow{\varphi^n} & & \xrightarrow{\varphi^n} \\
 & & \text{Ker } \varphi^n \xrightarrow{\cong} M \xrightarrow{\cong} \text{Im } \varphi^n & & \square
 \end{array}$$

This is a splitting of M

Exercise 2

$$* Q = \begin{Bmatrix} & 0 \\ 1 & & \\ & 2 \end{Bmatrix}$$

We start by finding all the almost split seq.

$$0 \rightarrow 0 \xrightarrow{K} K \xrightarrow{K} K \xrightarrow{K} K \xrightarrow{0} 0 \rightarrow 0$$

$$0 \rightarrow K \xrightarrow{K} K \xrightarrow{K} K \xrightarrow{0} 0 \xrightarrow{K} 0 \rightarrow 0$$

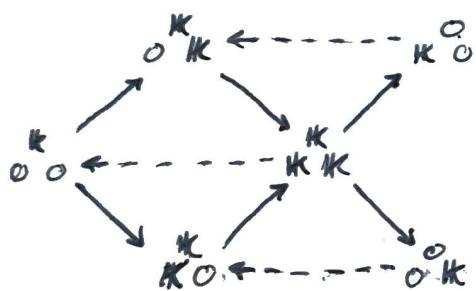
$$0 \rightarrow 0 \xrightarrow{K} K \xrightarrow{K} K \xrightarrow{K} K \xrightarrow{0} 0 \rightarrow 0$$

SI

$$K \xrightarrow{K} K \xrightarrow{0} 0 \xrightarrow{K} K$$

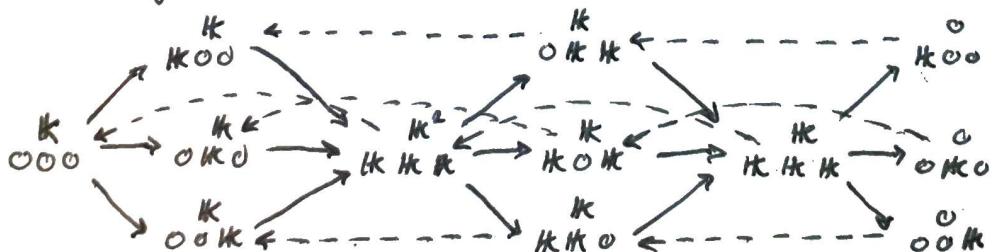
We know we also have
 $\text{rad } P \rightarrow P$ is right minimal
 almost split VP indec. proj.

↗ arrow from $\text{rad } P$ to P



$$* Q = \begin{Bmatrix} & 0 & \\ 1 & \downarrow & \\ & 2 & \end{Bmatrix}$$

Indec. proj. $KOO \xrightarrow{K} OKO \xrightarrow{K} OOK \xrightarrow{K} OOO$, $\text{rad} \rightsquigarrow OOO \& 0$



Exercise 3

M is indecomposable proj and inj.
WTS: $0 \rightarrow \text{rad } M \rightarrow M \oplus \frac{\text{rad } M}{\text{soc } M} \rightarrow \frac{M}{\text{soc } M} \rightarrow 0$
 is almost split exact.

Assume that M is non-simple. $\text{rad } M$ is left minimal
 right minimal almost split $\rightarrow \text{rad } M \rightarrow M$, $M \rightarrow \frac{M}{\text{soc } M}$ are irreducible.

$\text{rad } M$ is indec. as $\text{soc rad } M = \text{soc } M = \text{simple}$,
 $\frac{M}{\text{soc } M}$ is indec.

Almost split seq. starting in M .

$$0 \rightarrow \text{rad } M \rightarrow M \oplus ? \rightarrow \tau^-\text{rad } M \rightarrow 0$$

$\Rightarrow \exists$ irreduc. map $M \rightarrow \tau^-\text{rad } M$ \leftarrow All of irreducible out of M
 $\Rightarrow \tau^-\text{rad } M \cong \frac{M}{\text{soc } M}$ \leftarrow Use indec.

We have a square:

$$\begin{array}{ccccc} \text{rad } M & \longrightarrow & M & \longrightarrow & \frac{M}{\text{rad } M} \\ \downarrow & & \downarrow & & \downarrow \text{SI} \\ ? & \longrightarrow & \frac{M}{\text{soc } M} & \longrightarrow & \frac{M}{\text{rad } M} \end{array} \quad \begin{matrix} \text{Characterization of pushout} \\ \text{and pullbacks} \end{matrix}$$

$$\Rightarrow ? \cong \frac{\text{rad } M}{\text{soc } M}$$

Noethers isomorphism lemma

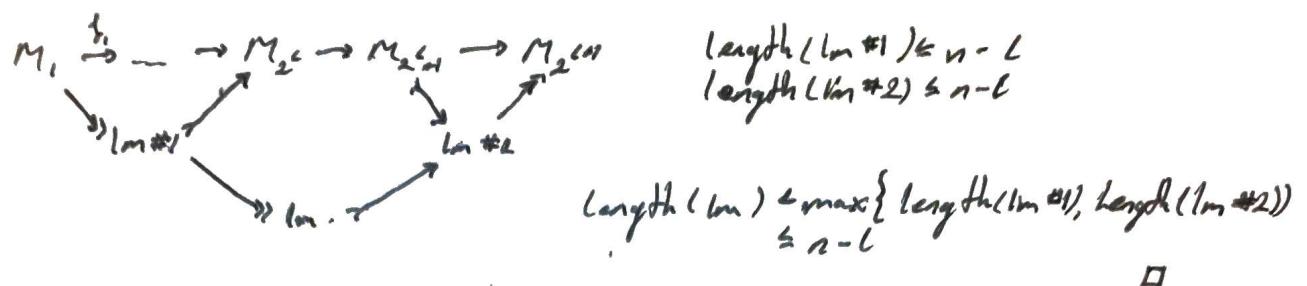
Haneda - Sai lemma

$M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} \dots \xrightarrow{f_{l-1}} M_{l+1}$ is a sequence of non-isomorphisms between indecomposables of length at most n . Then the image of the composition has length at most $n-l$ or 0. In particular, for $l > n$, the image of the composition is 0.

Pf. By induction on l

$L = l$:
 $M_1 \xrightarrow{f_1} M_2$ is not an iso. $\text{Length}(\text{Im } f_1) \leq \max\{\text{length } M_1, M_2\} \leq n$

$$l \approx l+1$$



□

Theorem

Let A be a f.d. algebra. Assume M is a finite set of indecomposable module such that:

- * Every indec. proj. is in M up to iso
- * M is closed under irreducible maps

$f: M \rightarrow N$ irred. and $M \in M \Rightarrow N \in M$
Then every indec. module is in M .

pf

Assume this is not true, i.e. M indec. s.t. $M \notin M$.

$\exists M_i$ indec. proj. s.t. $\text{Hom}(M_i, M) \neq 0$. $M_i \neq M_j \Leftrightarrow M_i \in M$.
Pick $M_i \rightarrow E_i$ as left minimal almost split.

$M_i \rightarrow E_i$ \exists indec summand M_2 of E_i s.t. $M_i \rightarrow M_2 \rightarrow M \neq 0$
 $\text{or } \downarrow$ where $M_i \rightarrow M_2$ is irreducible. Thus $M_2 \in M$, and
 $M_2 \neq M$.

Pick $M_2 \rightarrow E_2$ as left minimal almost split. We iterate the construction. \exists indec summand M_3 of E_2 s.t.
 $M_2 \rightarrow M_3 \rightarrow E_2$ the composition $M_i \rightarrow M_2 \rightarrow M_3 \rightarrow E \neq 0$, and $M_3 \in M$.

\downarrow This gives us an arbitrary sequence.

$M_i \rightarrow M_2 \rightarrow \dots \rightarrow M_n \rightarrow M \neq 0$ of any length l .

Since the dimension of M is bounded, pick $l >$.

Harada-Sai
 \Rightarrow The composition is 0 \Downarrow □

End of Lecture ~ Start of lecture

Brouer - Thrall conjecture

Infinite many indec.
up to iso

A is an algebra of infinite representation type.

1. There are arbitrary large indec. arbitrary dim.
2. There are dimensions d s.t. there are infinitely many d -dimensional indec.

Remark

1. is proven for K being algebraically closed, but this is based on a lot of new theory.

Hence we only prove 1.

Pf:

Assume that 1. is not true. Then there is an upper bound for the dimension of the indec. For M indec. $\exists P$ indec. proj. with $P \xrightarrow{\cong} M$. M could be P , but there are only finitely many, so we can pick $M \neq P$. This map factors through:

$$\begin{array}{ccc} P & \xrightarrow{\text{left adj}} & X_1 \\ & \downarrow & \downarrow \\ & M & \end{array}$$

$$\begin{array}{ccc} Y_1 & \xrightarrow{\text{left adj}} & X_2 \\ & \downarrow & \downarrow \\ & M & \end{array}$$

Then \exists indec. summand Y_i of X_i s.t.

$P \rightarrow Y_i \rightarrow X_i$ is non-zero. Y_i could be 0 , but there are only finitely many choices, or $Y_i \neq M$, then we have another factorization.

Continuing this forever we get either some finite list of indec. or a seq. $P \rightarrow Y_1 \rightarrow \dots \rightarrow Y_d \rightarrow \dots \rightarrow M$. If d is the upper bound of dimension, then by Hara Bi-Sai, the seq. is 0 \square

Abelian enriched category

Radical

Let \mathcal{A} be a pre-additive category.

$\text{rad } \mathcal{A}$ = intersection of all max right ideals.

Note as right ideals $\mathcal{A} = \bigoplus_{\text{objects}} \mathcal{A}(-, A)$.

We want to prove this

$$(\text{rad } \mathcal{A})(-, A) = \{ f: B \rightarrow A \mid \forall g: A \rightarrow B \text{ s.t. } 1 - fg \text{ is iso} \}.$$

A ^{right} ideal of a category is a collection of maps for every object $A, B: I(A, B)$ s.t. $I(A, B)$ is closed under addition, and $I(A, B) \times \mathcal{A}(A', A) \subseteq I(A', B)$ by composition on the right.

$\text{rad } \mathcal{A}$ is a right ideal, so $\text{rad } \mathcal{A}(A, B) \subseteq \mathcal{A}(A, B)$

Pf

Pick $f \in \text{rad} \mathcal{A}(B, A) \rightsquigarrow f$ is contained in every max right sub ideal of $\mathcal{A}(-, A)$. Then fg is in every max right sub ideals of $\mathcal{A}(-, A)$. So $\text{id}_A - fg$ is not contained in any max right sub ideal. So $\text{id}_A - fg$ is invertible. There is an h s.t. $(\text{id}_A - fg)h = \text{id}_A$
 $\Leftrightarrow \text{id}_A - fgh = h$ is right invertible.
 h is invertible with $\text{id}_A - fg$ as. inv.

Suppose $1-fg$ is an iso $\forall g: A \rightarrow B$. Assume there is a maximal subideal M of $\mathcal{A}(-, A)$ not containing f .
 $M + f \mathcal{A}(-, B) = \mathcal{A}(-, A)$
 $\uparrow \quad \uparrow \quad \curvearrowright$
 $m + fg = \text{id}_A$ Thus $m = \text{id}_A - fg$ is an iso \square

Then

At a preadditive cat, $f: A \rightarrow B$ in \mathcal{A} . TFAE:

1. $f \in \text{rad } \mathcal{A}(A, B)$
2. $\forall g: B \rightarrow A \quad \text{id}_B - gf$ is iso
- 2'. $f \in (\text{rad}(\mathcal{A}^{\text{op}}))^{\text{op}}(A, B)$
- 2'' $\forall g: B \rightarrow A \quad \text{id}_B - fg$ is iso

Cor

If preadditive, $A \in \mathcal{A}$, then $\text{rad } \mathcal{A}(A, A) \cong \text{rad } \text{End}_A(A)$

Warning

Typically $\text{rad}^2 \mathcal{A}(A, A) \neq \text{rad}^2 \text{End}_A(A)$ this is the square of radicals

Prop

Let \mathcal{A} be Krull-Schmidt additive.

An additive category w/
the Krull-Schmidt thm.

$f: \bigoplus A_i \rightarrow \bigoplus B_j$, where A_i, B_j are indec.

Then f is in $\text{rad} \mathcal{A}$ if and only if non of the components $f_{ij}: A_i \rightarrow B_j$ is an iso.

Pf

The radical does not contain any isomorphism, thus we have " \Rightarrow ".

" \Leftarrow " It is enough to show that $f_{ij}: A_i \rightarrow B_j$ is in $\text{rad} \mathcal{A}$.

For any $g: B_j \rightarrow A_i$, gf_{ij} is still not an iso. This means that gf_{ij} is in the maximal ideal of $\text{End}_A(A) \Rightarrow \text{id}_A - gf_{ij}$ is iso. \square

May be generalized to Krull-Schmidt categories

Prop

Let A be a fd. algebra. M, N are indec. modules.
Then $f: M \rightarrow N$ is irreducible if and only if
 $f \in \text{rad mod}^A(M, N)$, but $f \notin \text{rad}^2 \text{mod}^A(M, N)$.

Pf-

$f \in \text{rad mod}^A(M, N) \Leftrightarrow f \text{ not iso} \Leftrightarrow f \text{ neither split-mono}$
nor split-epi.

$f \in \text{rad}^2 \text{mod}^A(M, N) \Leftrightarrow f = gh \text{ where } g, h \in \text{rad mod}^A$.
so f cannot be irreducible.

If f is not irreducible, then $f = gh$ w/ h not split-mono
nor g split-epi. Since $h: \text{indec} \rightarrow ?$, not split-mono
 \Leftrightarrow being in the radical. $g: ? \rightarrow \text{indec}$, not split-epi
 \Leftrightarrow being in the radical. \square

What's about to come.

There is an arrow in AR-quiver \Leftrightarrow we have f as above.
To measure the size of irreducible maps we will be looking at
the quotient of $\text{rad mod}^A(M, N) / \text{rad}^2 \text{mod}^A(M, N)$