

Reptori - Week 7

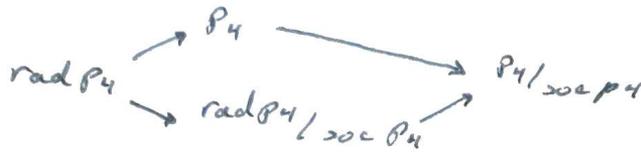
Exercises

1. $\mathbb{K} \left\{ \begin{array}{ccc} 1 & \rightarrow & 2 \\ \downarrow & & \downarrow \\ 3 & \rightarrow & 4 \end{array} \right\} / \langle ca - db \rangle$

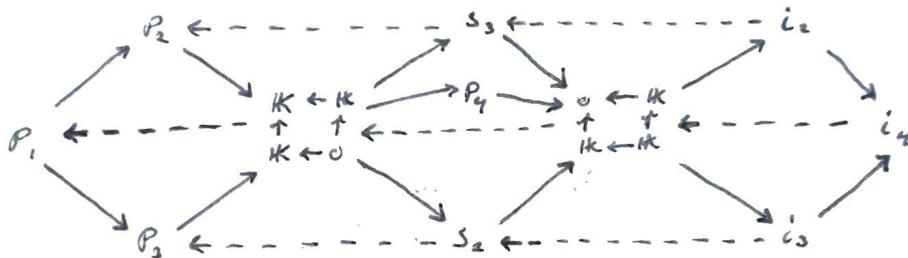
These are the indec. projectives.

$P_1 = \begin{array}{c} \mathbb{K} \leftarrow 0 \\ \uparrow \\ 0 \leftarrow 0 \end{array}$, $P_2 = \begin{array}{c} \mathbb{K} \leftarrow \mathbb{K} \\ \uparrow \\ 0 \leftarrow 0 \end{array}$, $P_3 = \begin{array}{c} \mathbb{K} \leftarrow 0 \\ \uparrow \\ \mathbb{K} \leftarrow 0 \end{array}$, $P_4 = \begin{array}{c} \mathbb{K} \leftarrow \mathbb{K} \\ \uparrow \\ \mathbb{K} \leftarrow \mathbb{K} \end{array}$ → this is also injective

We use the exercise from last week to find a mesh:



$\text{rad } P_4 \cong \begin{array}{c} 0 \leftarrow \mathbb{K} \\ \uparrow \\ \mathbb{K} \leftarrow 0 \end{array}$
 $\cong \begin{array}{c} 0 \leftarrow \mathbb{K} \\ \uparrow \\ 0 \leftarrow 0 \end{array} \oplus \begin{array}{c} 0 \leftarrow 0 \\ \uparrow \\ \mathbb{K} \leftarrow 0 \end{array}$
 S_2 \nearrow S_3

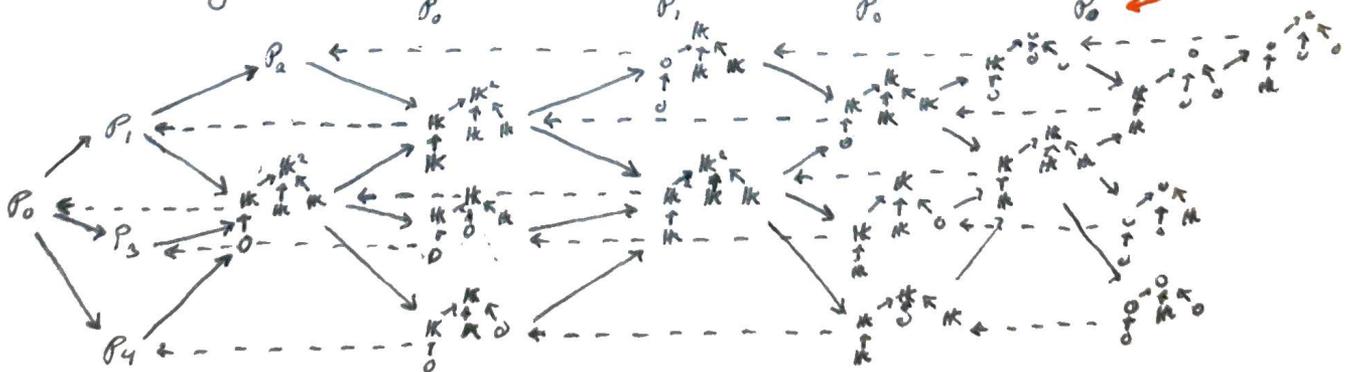


2. $\mathbb{K} \left\{ \begin{array}{ccc} 1 & \leftarrow & 0 \\ \downarrow & & \downarrow \\ 2 & & 3 \\ & & \downarrow \\ & & 4 \end{array} \right\}$

$P_0 = \begin{array}{c} \mathbb{K} \leftarrow \mathbb{K} \\ \uparrow \\ 0 \leftarrow 0 \end{array}$, $P_1 = \begin{array}{c} \mathbb{K} \leftarrow \mathbb{K} \\ \uparrow \\ 0 \leftarrow 0 \end{array}$, $P_2 = \begin{array}{c} \mathbb{K} \leftarrow \mathbb{K} \\ \uparrow \\ 0 \leftarrow 0 \end{array}$, $P_3 = \begin{array}{c} \mathbb{K} \leftarrow \mathbb{K} \\ \uparrow \\ 0 \leftarrow 0 \end{array}$, $P_4 = \begin{array}{c} \mathbb{K} \leftarrow \mathbb{K} \\ \uparrow \\ 0 \leftarrow 0 \end{array}$

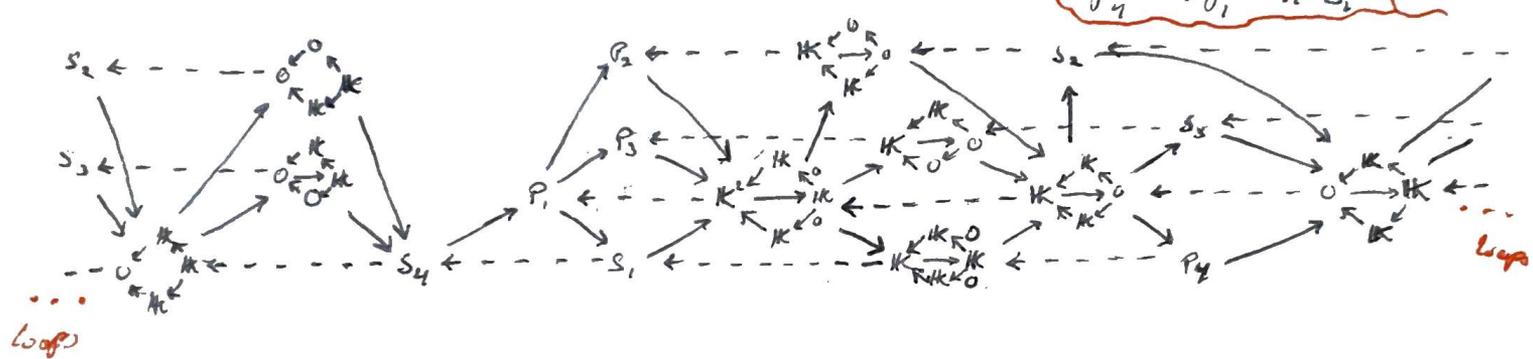
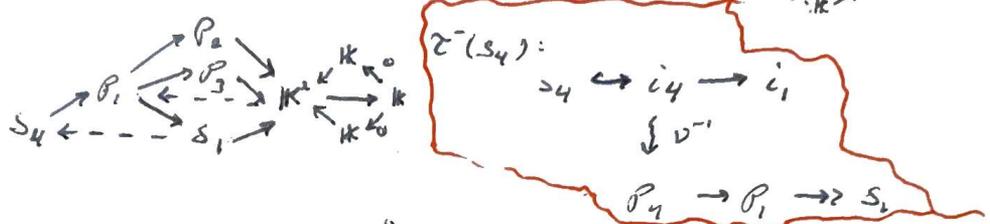
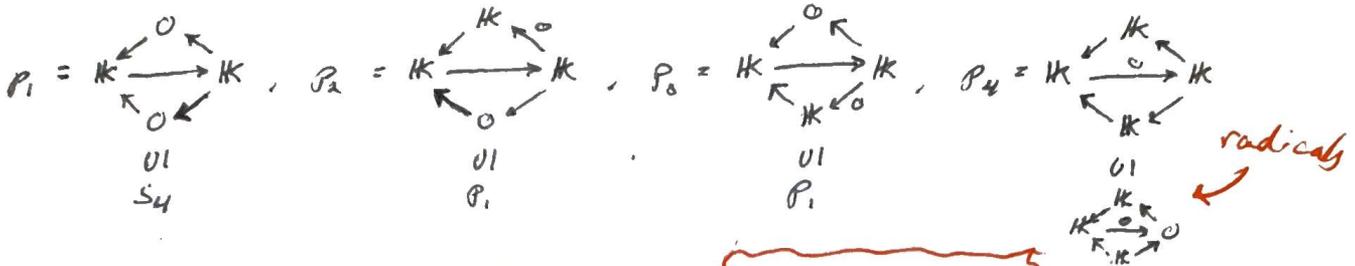
projectives

radicals



$$3. \mathbb{K} \left\{ \begin{array}{ccc} 1 & \begin{array}{cc} \xrightarrow{a} 2 & \xrightarrow{b} 4 \\ \xleftarrow{e} & \\ \xrightarrow{c} 3 & \xrightarrow{d} 4 \end{array} & \end{array} \right\} / \langle eb, ed, ba-dc, baed \rangle$$

indecomproj



Prop Let M, N be indec. and there is a left minimal almost split map $M \xrightarrow{f} N^a \oplus R$, where N is not a summand of R .

$\begin{pmatrix} f_1 \\ \vdots \\ f_a \end{pmatrix}$ Then the images of f_1, \dots, f_a form a basis of $\text{rad}(M, N) / \text{rad}^2(M, N)$ as $(\text{End} N / \text{rad} \text{End} N) - \text{v.s.}$

Pf - generating: Pick $f \in \text{rad}(M, N) \rightarrow f$ not split mono, so f factors through the map above:

$$f = \alpha_1 f_1 + \dots + \alpha_a f_a + \beta g \text{ for } \alpha_i \in \text{End} N, \beta \in \text{Hom}(R, N).$$

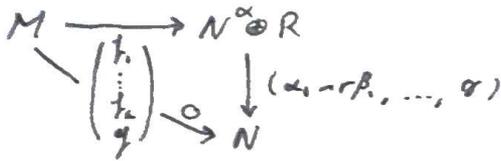
$$\underbrace{\hspace{10em}}_{\beta g \in \text{rad}^2 \text{ mod } \text{rad}^2} \Rightarrow f = \sum \alpha_i f_i \text{ mod } \text{rad}^2$$

Note that α_i may be chosen from $\text{End} N / \text{rad} \text{End} N$ whenever we consider f mod rad^2 .

End of lecture ~ Start of Lecture

linear independence: Let $\alpha_1 f_1 + \dots + \alpha_n f_n = 0$ in mod rad^2 ,
 i.e. $\sum \alpha_i f_i \in \text{rad}^2$. Since it is in the radical it is $r s_i$ for $r, s_i \in \text{rad}$.
 Since $s \in \text{mod rad}^A$, it factors as $s = (\beta_1, \dots, \beta_n, \gamma) \begin{pmatrix} f_1 \\ \vdots \\ f_n \\ y \end{pmatrix}$
 $= \beta_1 f_1 + \dots + \beta_n f_n + \gamma y$.

Now $\sum \alpha_i f_i = r(\beta_1 f_1 + \dots + \beta_n f_n + \gamma y)$
 $\Rightarrow (\alpha_1 - r\beta_1) f_1 + \dots + (\alpha_n - r\beta_n) f_n + \gamma y = 0$



Minimality of $\begin{pmatrix} f_1 \\ \vdots \\ f_n \\ y \end{pmatrix}$ implies that $(\alpha_1 - r\beta_1, \dots, \alpha_n, \gamma)$ is not split epi.
 \Rightarrow none of $\alpha_i - r\beta_i$ are iso
 $\text{rad} \subset r \subset \text{rad}$

\Rightarrow none of α_i are iso
 \Rightarrow All of α_i are in $\text{rad End}(N)$

These α_i are zero when considered as a $(\text{End } N / \text{rad End } N)^{\text{op}}$ vs. \square

Corollary

If $M \rightarrow N^{\oplus n} \oplus R$ is left minimal almost split, M, N indec, N not a direct summand of R .

Then $a = \frac{\dim_K (\text{rad}(M, N) / \text{rad}^2(M, N))}{\dim_K (\text{End}(N) / \text{rad End}(N))}$

Corollary

Let M, N be indec. Assume there is an arrow $M \rightarrow N$ in AR -quiver. Then the label on this arrow is

$\frac{\dim_K (\text{rad End}(M, N) / \text{rad}^2(M, N))}{\dim_K (\text{End } M / \text{rad End } M)}, \frac{\dim_K (\text{rad}(M, N) / \text{rad}^2(M, N))}{\dim_K (\text{End } N / \text{rad End } N)}$

Corollary

If K is algebraically closed, then the denominators are both 1, and the 2 arrows coincide.

Remark

A typical example with different labels

$$\begin{pmatrix} R & e \\ 0 & e \end{pmatrix}$$

Corollary

$M \xrightarrow{(a, \tilde{a})} N$ in AR -quiver, with M not injective, then we have the arrow $N \xrightarrow{(b, \tilde{b})} \tilde{M}$

Pf

Already know $\exists N \xrightarrow{(b, \tilde{b})} \tilde{M}$. $\tilde{a} = b \frac{\dim_K \text{End } N / \text{rad End } N}{\dim_K \text{End } \tilde{M} / \text{rad End } \tilde{M}}$

$$a = l \cdot \frac{\dim_k \text{End}(N) / \text{rad End}(N)}{\dim_k \text{End}(M) / \text{rad End}(M)}$$

We know that $\text{End } M / \text{rad End } M \cong \text{End } \tilde{M} / \text{rad End } \tilde{M}$,
so $a = \tilde{a}$ \square

Rem

A knitting of the AR-quiver works in general.
If we know all arrows into a given indecomposable M ,
then obtain all of the arrows out of the same indec. as:

- $M \xrightarrow{\text{lar}(f)} N$ for every $N \xrightarrow{g} M$, N not inj.
- $M \xrightarrow{\text{lar}(g)} P$, whenever $M \in \text{rad } P$ as a direct summand.

Nakayama algebras

Def.

A module is called uniserial if it has a unique composition series, up to **EQUALITY** is used...

Obs

M uniserial \Leftrightarrow submodules of M are linearly ordered by inclusion.
 $\Leftrightarrow \{\text{rad}^m M\}_{m \in \mathbb{N}}$ are every submodules of M .

\hookrightarrow If M is uniserial, it has a unique maximal submodule, which is the radical. The maximal submodule is still uniserial, so we do recursion to obtain every submodule $\{\text{rad}^m M\}_{m \in \mathbb{N}}$.

If M is linearly ordered (submods), then this ordering is clearly the composition series. This also makes up the radical filtration.

Obs

M uniserial $\Leftrightarrow \text{rad}^m M / \text{rad}^{m+1} M$ is simple or $0 \forall m$

\hookrightarrow

" \Rightarrow " This is clear

" \Leftarrow " The radical filtration is a composition series. Moreover, by definition the radical is the unique maximal submodule. This is then the only composition series.

Def.

An algebra A is called Nakayama if the indecomposable projectives and -injectives are uniserial

Lemma

$A = \mathbb{K}Q/\langle R \rangle$. Every indec. projectives are uniserial
 \Leftrightarrow there is at most one arrow ending in every vertex.

Pf

" \Rightarrow " Let i be a vertex. $P_i =$ paths ending in i .

$\alpha \rightarrow \beta$ \langle paths ending in α \rangle \langle paths ending in β \rangle are incompatible. So this goes against uniseriality.

" \Leftarrow " Assume there is at most one arrow into every vertex. Then there is at most one arrow of length 2 into every vertex. Moreover, there is at most one arrow of length n into every vertex.

$$\text{rad}^n(P_i) / \text{rad}^{n+1}(P_i) = \frac{\langle \text{paths of length } n \text{ ending in } i \rangle}{\langle \text{paths of length } n \text{ not ending in } i \rangle}$$

which is simple. So every projective have the radical filtration as composition series.

Corollary

$A = \mathbb{K}Q/\langle R \rangle$ is Nakayama.

$\Leftrightarrow Q$ is a disjoint union of quivers on the form:



For $A = \mathbb{K}Q/\langle R \rangle$ Nakayama, we write $M_i^l = P_i / \text{rad}^l P_i$. This is uniserial of length l .

Prop

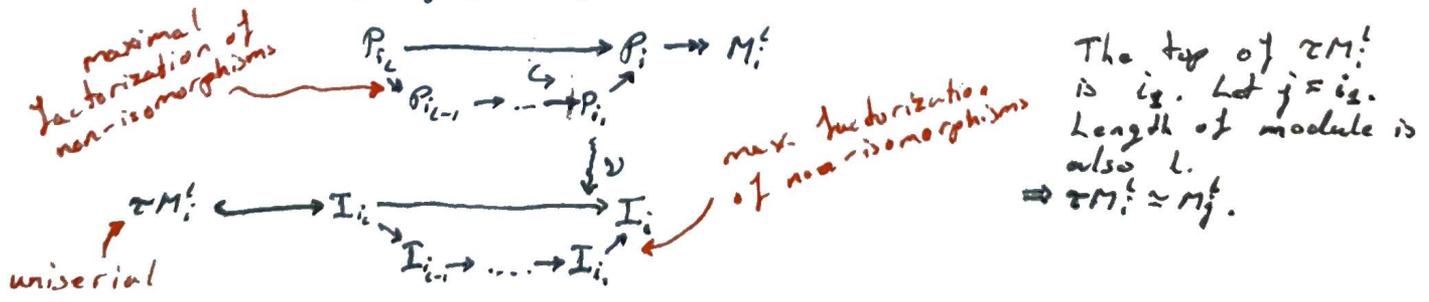
if M_i^l is not projective, then there is an arrow $j \rightarrow i$ in Q
 $\text{st. } \tau M_i^l = M_j^l$

these arrows gen rad^l .

Pf

$\text{rad}^l P_i \neq 0$ $i_l \rightarrow \dots \rightarrow i_1 \rightarrow i$ in Q is non-zero in $\mathbb{K}Q/\langle R \rangle$.

Pick a proj pres of M_i^l :



The top of τM_i^l is i_2 . Let $j \neq i_2$. Length of module is also l .
 $\Rightarrow \tau M_i^l \cong M_j^l$.