Repteori - Week 8

Exercises Find the AR-quiver of:

1) $\begin{pmatrix} \mathbb{R} & \mathbb{C} \\ 0 & \mathbb{C} \end{pmatrix} = A$

$P_1 = (\mathbb{R} \;\; \mathbb{C})$ and $\overset{P_2}{(0 \;\; \mathbb{C})}$ are right modules, and
This is simple

$(\mathbb{R} \;\; \mathbb{C}) \oplus (0 \;\; \mathbb{C}) \simeq A$, so they are projective.

$\operatorname{rad} A = \begin{pmatrix} 0 & \mathbb{C} \\ 0 & 0 \end{pmatrix}$ this is an ideal, nilpotent and quotient is s.s, i.e. $\mathbb{R} \times \mathbb{C}$

$\operatorname{rad} P_1 = P_1 \operatorname{rad} A = (0 \;\; \mathbb{C})$

$P_1 / \operatorname{rad} P_1$ is simple and P_1 is indec.

$$\begin{tikzcd}
 & P_1 \arrow[dr, "{(2,1)}"] & & (\mathbb{R} \;\; 0) \arrow[ll, dashed] \\
P_2 \arrow[ur, "{(1,2)}"] & & (\mathbb{C} \;\; \mathbb{C}) \arrow[ll, dashed] \arrow[ur, "{(1,2)}"] &
\end{tikzcd}$$

$\operatorname{Hom}(P_2, P_1) \simeq \mathbb{C}$

$\operatorname{End}(P_2) \simeq \mathbb{C}$

$\operatorname{End}(P_1)/\operatorname{rad}\operatorname{End}(P_1) \simeq \operatorname{End}(P_1/\operatorname{rad} P_1)$

$\simeq \mathbb{R}$

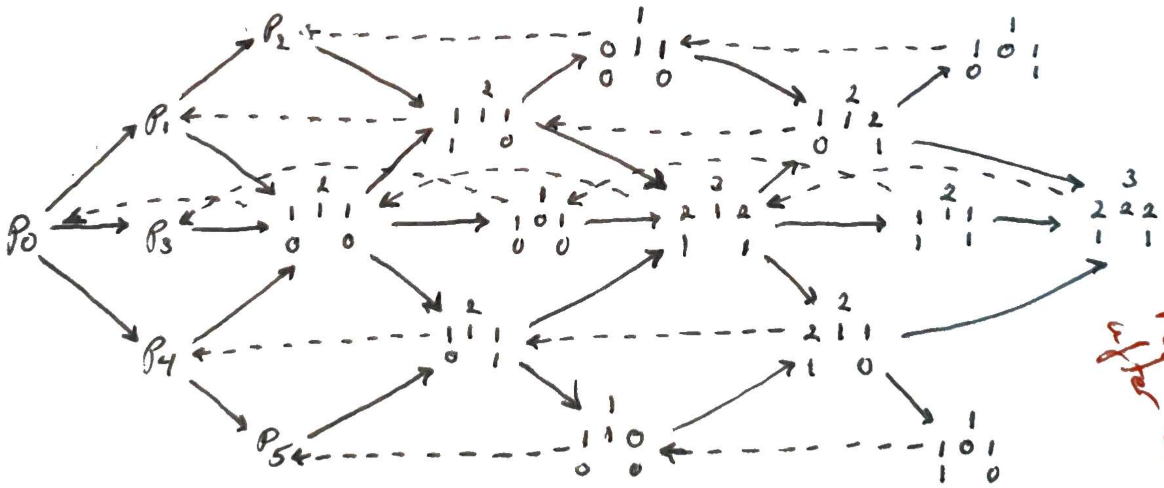
2) $\mathbb{K}\left\{ \begin{smallmatrix} & 0 & \\ \swarrow & \downarrow & \searrow \\ 1 & 3 & 4 \\ \downarrow & & \downarrow \\ 2 & & 5 \end{smallmatrix} \right\}$

These are called dim. vector

indec. proj. $P_0 = \begin{smallmatrix} &1& \\ 0&0&0 \\ 0&&0 \end{smallmatrix}$, $P_1 = \begin{smallmatrix} &1& \\ 1&0&0 \\ 0&&0 \end{smallmatrix}$, $P_2 = \begin{smallmatrix} &1& \\ 1&0&0 \\ 1&&0 \end{smallmatrix}$, $P_3 = \begin{smallmatrix} &1& \\ 0&1&0 \\ 0&&0 \end{smallmatrix}$, $P_4 = \begin{smallmatrix} &1& \\ 0&0&1 \\ 0&&0 \end{smallmatrix}$, $P_5 = \begin{smallmatrix} &1& \\ 0&0&1 \\ 0&&1 \end{smallmatrix}$

$\cup$ $\cup$ $\cup$ $\cup$ $\cup$ $\cup$

radicals P_0, P_0, P_1, P_0, P_0, P_4

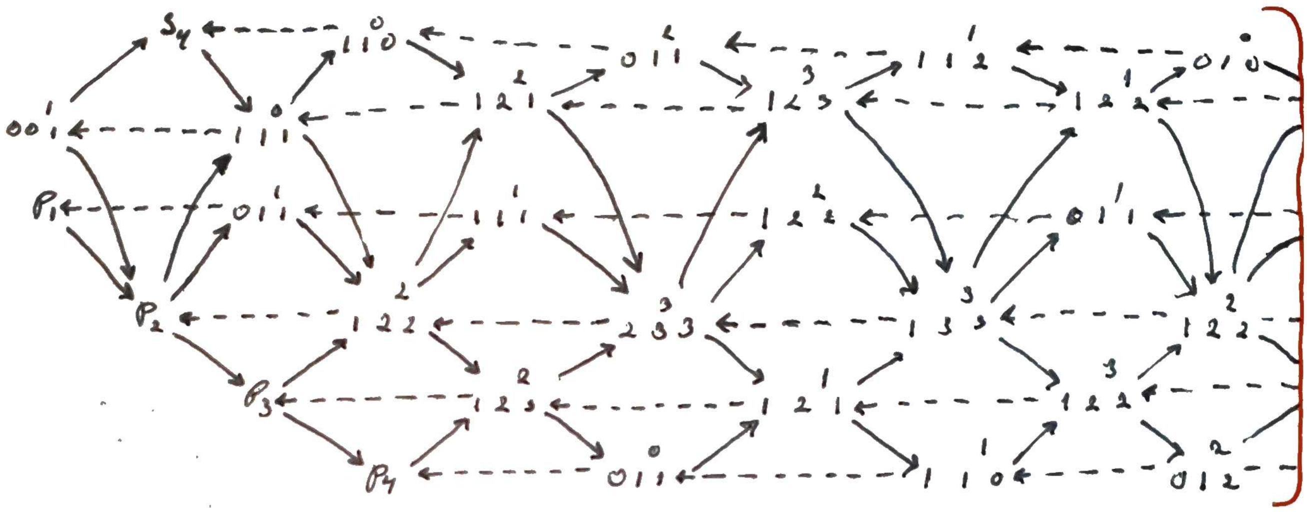


And it goes on.......

Until it ends. This is of finite representation type after all.

3) $\mathbb{K}\{1 \xrightarrow{a} 2 \xrightarrow{b} 3 \xrightarrow{c} 4 \xrightarrow{d} 2\}/\langle dcb\rangle$

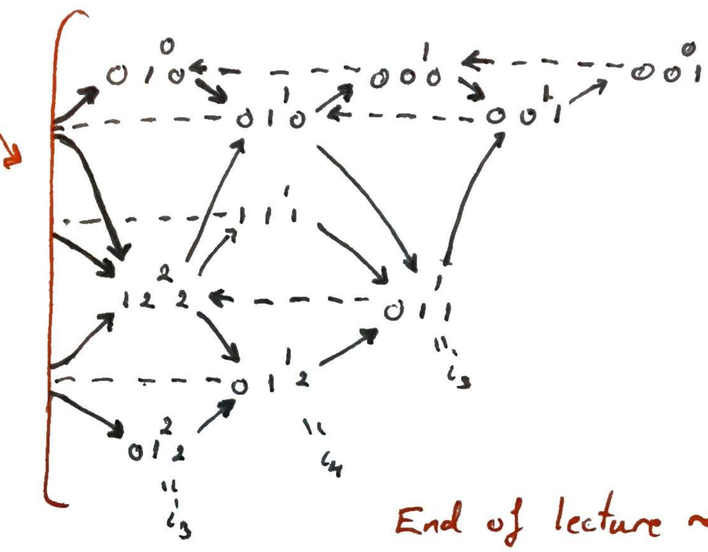
indec. proj.	$P_1 = 1\,0\,{}^{0}\,0$,	$P_2 = 1\,1\,{}^{1}\,1$,	$P_3 = 1\,1\,{}^{2}\,1$,	$P_4 = 1\,1\,{}^{2}\,2$
	$\cup\!\shortmid$	$\cup\!\shortmid$	$\cup\!\shortmid$	$\cup\!\shortmid$
radicals	P_1	$1\,0\,{}^{1}\,1 \simeq P_1 \oplus 0\,0\,{}^{1}\,1$	P_2	P_3



Calculating τ^- in order to start.

$\tau^-(0\,0\,{}^{1}\,1) = \operatorname{cok}(\nu^{-1}(i_3) \to \nu^{-1}(i_2)) = \operatorname{cok}(P_3 \to P_2) = 1\,1\,{}^{0}\,1$

$\tau^-(S_4) = \operatorname{cok}(\nu^{-1}(i_4) \to \nu^{-1}(i_2)) = \operatorname{cok}(P_4 \to P_2) = 1\,1\,{}^{0}\,0$



End of lecture ~

Remember Nakayama algebras from last week.
We proved that $\tau M_i^\ell = M_j^\ell$ where $j \to i$ if M_i^ℓ is not projective.

Prop.
In the set up above, the almost split seq. ending in M_i^ℓ is $0 \to M_j^\ell \to M_j^{\ell-1} \oplus M_i^{\ell+1} \to M_i^\ell \to 0$.

Fact
$0 \to \tau M \to E \to M \to 0$ is non-split, then it is almost split if any $\varphi \in \mathrm{End}(M) \setminus \mathrm{Aut}(M)$ factors throug E.

↳ $0 \to \tau M \to E \to M \to 0$ is our seq. Pick
$0 \to \tau M \to F \to M \to 0$ as an almost split seq.

$$\begin{tikzcd}
0 \arrow[r] & \tau M \arrow[r] & E \arrow[r] & M \arrow[r] & 0 \\
0 \arrow[r] & \tau M \arrow[r] \arrow[u, equal] & F \arrow[r] \arrow[u] & M \arrow[r] \arrow[u, "c"'] \arrow[ul, dashed] & 0
\end{tikzcd}$$

If c is iso we are done.
If c is not iso, then it factors through E. Since the right square is a pullback, the map $F \to M$ is split-epi ↯

Pf.

$$\begin{tikzcd}
M_j^\ell \arrow[r, hook] \arrow[d, two heads] \arrow[dr, phantom, "\circlearrowleft"] & M_i^{\ell+1} \arrow[d, two heads] \\
M_j^{\ell-1} \arrow[r, hook] & M_i^\ell
\end{tikzcd}$$

We have the following bicartesian square. It is bicartesian by the characterization of pushouts and pullbacks.

Pick any $\varphi \in \mathrm{End}(M_i^\ell) \setminus \mathrm{Aut}(M_i^\ell)$, so it is not surjective. So it factors through the maximal submodule, which is $M_i^{\ell-1}$. □

Thm
Let $A = \Bbbk Q/\langle R \rangle$ be a Nakayama algebra. Then $\{M_i^\ell\}$ are the indec. A-modules, up to isomorphism. In particular there are only finitely many indecomposables.

Pf.
We know that $\{M_i^\ell\}$ contain every indec. proj. For any M_i^ℓ, the left minimal almost split map starting in M_i^ℓ is

not inj → $M_i^\ell \to M_i^{\ell-1} \oplus M_j^{\ell+1}$ for some j

inj → $M_i^\ell \to M_i^{\ell-1}$ ← Also indec.

□

Cartan matrices

Def
A Cartan matrix is a square and integral matrix C s.t.:
* $c_{i,i} = 2$
* $c_{i,j} \le 0$ for $i \neq j$
* It is symmetrizable (∃ D diagonal invertible s.t. DC is symmetric)

We focus on the case which is symmetric.

Obs
A Cartan matrix can be depicted as a valued graph, w/ vertices $\{1, \dots, n\}$. There is an edge i — j if $c_{ij} < 0$, and it has value $-c_{ij} = -c_{ji}$.

Def
A Cartan matrix is called indec. if the graph is connected.

Def
C is of finite type if C is positive definite, i.e. $x^T C x > 0 \ \forall x \in \mathbb{R}^n \setminus \{0\}$

C is of affine type if C is semi-positive definite, i.e. $x^T C x \geq 0 \ \forall x \in \mathbb{R}^n$

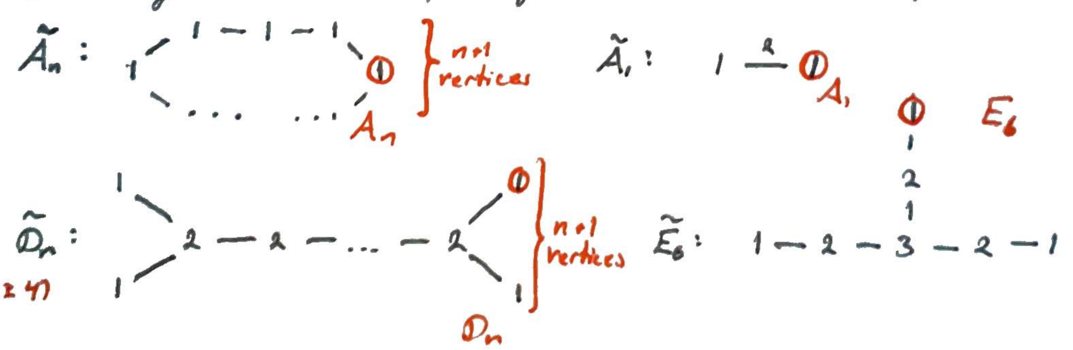
Note
$e_i^T C e_i = 2 \Rightarrow C$ can either be positive, semi-positive or indefinite

Def
This + is maybe too strict

for a diagram depiction, additive means $2 \times$ # in vertex $= \sum$ of # in neighbours

A map $\{1, \dots, n\} \xrightarrow{f} \mathbb{N}^+$ is called subadditive if $\forall i \ \sum_j c_{ij} f(j) \geq 0$. It is additive if every such term is 0, i.e. $Cf = 0$.

The following is a list of diagrams with additive functions:



$$\begin{array}{l}
\qquad\qquad\qquad\;\; 2 \\
\qquad\qquad\qquad\;\; | \\
\tilde{E}_7 : \; 1 - 2 - 3 - 4 - 3 - 2 - \textcircled{1} \qquad E_7
\end{array}$$

$$\begin{array}{l}
\qquad\qquad\qquad\qquad\qquad\quad\; 3 \\
\qquad\qquad\qquad\qquad\qquad\quad\; | \\
\tilde{E}_8 \;\; \textcircled{1} - 2 - 3 - 4 - 5 - 6 - 4 - 2 \qquad E_8
\end{array}$$

Remark

If we allowed non-symmetric labels, this list should be much longer.

Def.

These are called Euclidean diagrams. The Dynkin diagrams are the subdiagrams of the Euclidean without the circled vertex.

Note

* Any proper subdiagram of a Euclidean diagram is a disjoint union of Dynkin diagrams.
* Any connected graph is either Euclidean, properly contained in a Euclidean or properly contains a Euclidean
  This can be verified by brute force.

Prop.

Let C be a connected symmetric Cartan matrix, then its corresponding diagram is Euclidean if and only if there is an additive function
$\Leftrightarrow$ C is of affine type

Its corresponding diagram is Dynkin
$\Leftrightarrow$ there is a subadditive function, but not additive function
$\Leftrightarrow$ C is of finite type.

Pf.

We have seen that every Euclidean diagrams have additive functions. For Dynkin diagrams, we restrict the additive functions from the Euclidean, making a strictly subadditive function.
If a diagram is neither Euclidean nor Dynkin, then it contains a Euclidean diagram as a proper subgraph. If it admits an additive or subadditive function, then this function on the Euclidean diagram is strictly subadditive, call it g. Let f be the additive function in the Euclidean diagram.

$C_{Eu} f = 0$, $C_{Eu} g =$ vector of non-neg. numbers $\neq 0$.

$0 = (f^T C) g = f^T (C g) > 0$ ↯ g does not exist.