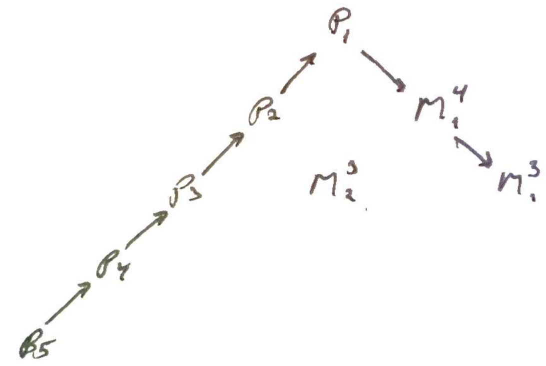
Repteori – Week 9

Exercise

Denne er Nakayama

1. $\mathbb{K}\{1\to 2\to 3\to 4\to 5\}$

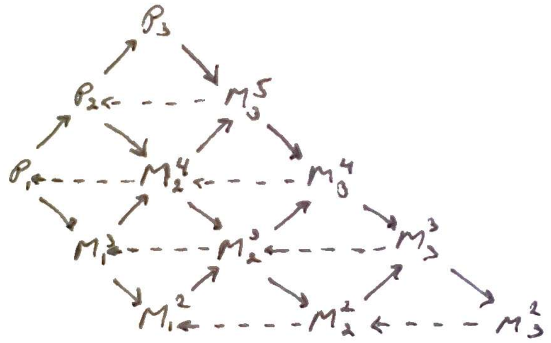
Look at thems



2. $\mathbb{K}\{1 \xrightarrow{a} 2 \xrightarrow{b} 3 \xrightarrow{c} 1\}/\{cabc\}$

This is also Nakayama

$P_1 \text{ "=" } M_1^4,\quad P_2 \text{ "=" } M_2^5,\quad P_3 \text{ "=" } M_3^6$

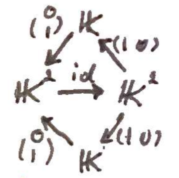
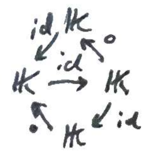


And more

3. $\mathbb{K}\{1 \xrightarrow{a} 2 \xrightarrow{b} 3,\ 3 \xrightarrow{e} 1,\ 1 \xrightarrow{c} 4 \xrightarrow{d} 3\}/\langle ab-dc,\ aeb,\ ced\rangle$

This is isomorphic to it's opposite

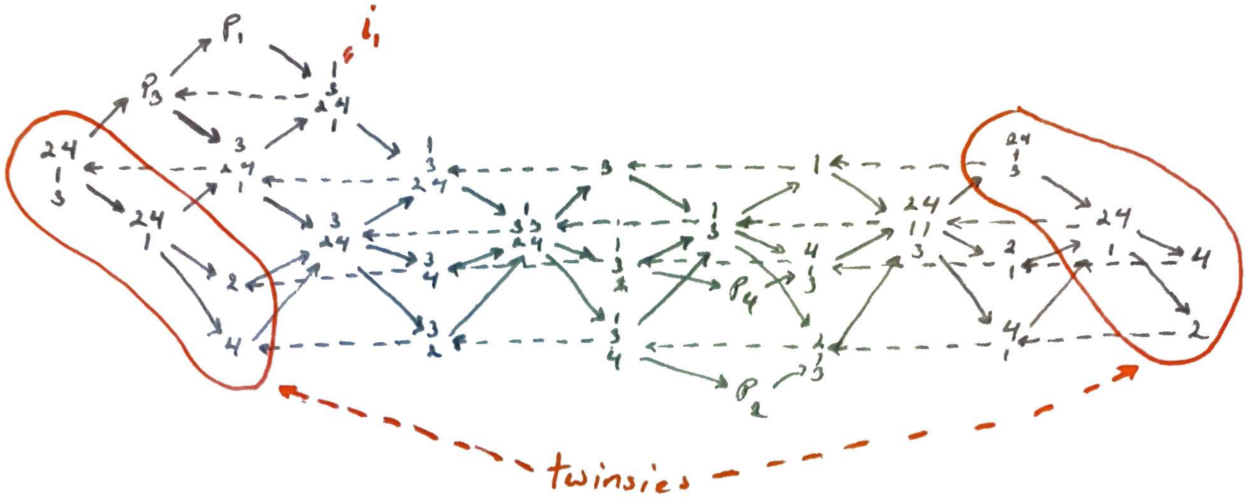
Projectives

$P_1 \overset{i_3}{\text{"="}} \begin{smallmatrix}1\\3\\2\,4\\1\\3\end{smallmatrix} = 2\,{}^1_1\,2 =$ , $\quad P_2 \overset{i_4}{\text{"="}} \begin{smallmatrix}2\\1\\3\\4\end{smallmatrix} = 1\,{}^1_1\,1 =$ 

$P_3 = \begin{smallmatrix}3\\2\,4\\1\\3\end{smallmatrix} = 1\,{}^1_1\,2,\quad P_4 \overset{i_2}{\text{"="}} \begin{smallmatrix}4\\3\\2\end{smallmatrix} = 1\,{}^1_1\,1 =$ 

Radical

$\operatorname{rad}P_1 = \begin{smallmatrix}3\\2\,4\\1\\3\end{smallmatrix},\quad \operatorname{rad}P_2 = \begin{smallmatrix}1\\3\\4\end{smallmatrix},\quad \operatorname{rad}P_3 = \begin{smallmatrix}2\,4\\1\\3\end{smallmatrix},\quad \operatorname{rad}P_4 = \begin{smallmatrix}1\\3\\2\end{smallmatrix}$



Continuing proof from last time

Let f be subadditive, wts. it is either positive definite or positive semi-definite.

$$x^t C x = \sum_{i,j} x_i C_{i,j} x_j = -\underbrace{\sum_{i,j} \frac{C_{i,j}}{2 f_i f_j} \overbrace{(f_j x_i - f_i x_j)^2}^{\text{Positive}}}_{\text{Positive}} + \underbrace{\sum \frac{C_{i,j}}{2 f_i f_j} (f_j^2 x_i^2 + f_i^2 x_j^2)}_{\text{Positive}}$$

$$\sum_{i,j} \frac{C_{i,j}}{2 f_i f_j} (f_j^2 x_i^2 + f_i^2 x_j^2)$$
$$= \sum_{i,j} \frac{C_{i,j}}{2 f_i} f_j x_i^2 + \sum_{i,j} \frac{C_{i,j}}{2 f_j} f_i x_j^2$$
$$= \sum_i \frac{1}{2 f_i} \underbrace{\left(\sum_j C_{i,j} f_j \right)}_{\text{Positive}} x_i^2 + \sum_j \frac{1}{2 f_j} \underbrace{\left(\sum_i C_{i,j} f_i \right)}_{\text{Positive}} x_j^2$$

$\Rightarrow$ C is positive semi-definite.

It is equal to 0, whenever:

- $f_j x_i = f_i x_j \ \forall i,j$ s.t. $C_{i,j} \neq 0$
  Since C is connected, this implies that all x_i are non-zero (or only 0)
- $x = 0$ or f is additive

If the diagram is neither Dynkin nor Euclidean, then it contains a Euclidean diagram as a subdiagram.

f add for Euclidean

We have 2 cases for indefinite

* more edges

$$f^T C f < f^T C_{add} f = 0$$

* more vertices, pick a vertex i which is not in the Euclidean subdiagram, but is a neighbour.

$$(2f + e_i)^T C (2f + e_i) = \underbrace{4 \underbrace{f^T C e_i}_{\leq 0}}_{\leq -4} + 2 < 0$$

We extend f to the entire diagram by adding zeros to the vertices outside of the Euclidean subdiagram.

$\Rightarrow C$ is indefinite $\square$

Self injective algebras representation finite

Def

A is self-injective if the projectives of A are the injectives.

Warning

ν does not have to be the identity

Examples

* $\mathbb{K}G$, where G is a finite group
* $\mathbb{K}\{1 \to 2 \to \cdots \to n \to 1\} / \langle \text{arrows} \rangle^l$
* $\mathbb{K}[x]/\langle x^l \rangle$

Note

$\underline{\text{mod}}_A = \overline{\text{mod}}_A$, so τ is an autoequivalence on $\underline{\text{mod}}_A$.

Rem

$\underline{\text{mod}}_A$ is a triangulated category.

Construction

Let Q be a (finite) quiver, possibly labeled like an AR-quiver.

Denote $\mathbb{Z}Q$ as the translation quiver which have
vertices: $\mathbb{Z} \times Q_0$
arrows: $(i, q) \xrightarrow{(a,b)} (i, q')$, $(i, q') \xrightarrow{(b,a)} (i+1, q)$ for arrows $q \xrightarrow{(a,b)} q' \in Q_1$
dashed arrows: $(i, q) \leftarrow \text{- - - - -} (i+1, q)$

Ex

$1 \xrightarrow{(1,2)} 2 \longrightarrow 3$

$$\begin{tikzcd}[column sep=small]
 & & & (0,3) \arrow[dr] & & (1,3) \arrow[ll, dashed] & & (2,3) \arrow[ll, dashed] & \\
\cdots & & (0,2) \arrow[ur] \arrow[dr, "(2,1)"] & & (1,2) \arrow[ur] \arrow[dr, "(2,1)"] \arrow[ll, dashed] & & (2,2) \arrow[ur] \arrow[ll, dashed] & & \cdots \\
 & (0,1) \arrow[ur, "(1,2)"] & & (1,1) \arrow[ur, "(1,2)"] \arrow[ll, dashed] & & (2,1) \arrow[ur, "(1,2)"] \arrow[ll, dashed] & & &
\end{tikzcd}$$

Lemma

$M_1 \overset{a}{\hookrightarrow} M_2 \overset{b}{\twoheadrightarrow} M_3$ short exact. Then

$\underline{\mathrm{Hom}}(N, M_1) \longrightarrow \underline{\mathrm{Hom}}(N, M_2) \longrightarrow \underline{\mathrm{Hom}}(N, M_3)$ is exact

Pf

The composition is zero.
Let $\varphi \in \mathrm{Hom}(N, M_2)$ s.t. $b\varphi$ factors through a proj.

$$\begin{tikzcd}
M_1 \arrow[r, hook, "a"] & M_2 \arrow[r, two heads, "b"] & M_3 \\
 & N \arrow[ul, dashed, "f"] \arrow[u, "\varphi"] \arrow[r, "c"'] & P \arrow[u, "d"'] \arrow[ul, dashed, "e"']
\end{tikzcd}$$

$\exists e$ s.t. $d = be$

$b(\varphi - ec) = 0$
so $\varphi - ec$ factors through f.

So $\varphi = af + ec \Rightarrow \varphi = af \in \underline{\mathrm{Hom}}(N, M_2)$ $\square$

End of lecture ~ Start of lecture

Thm

Λ self-inj and rep. finite, then the AR-quiver of $\underline{\mathrm{mod}}\,\Lambda$ is of the form $\mathbb{Z}Q/G$, where Q is a disjoint union of Dynkin diagrams, and G is a group of automorphisms of $\mathbb{Z}Q$ s.t. $\tau^n \in G$ for some n.

Pf

Since τ is an autoequivalence of $\underline{\mathrm{mod}}_\Lambda$, any component of the AR-quiver is of the form $\mathbb{Z}Q/G$. Q is some, possibly infinite, quiver. By representation finite, Q ought to be finite. $\mathbb{Z}Q/G$ having finitely many objects $\Leftrightarrow \tau^n \in G$ for some n.

Pick M to be any indec. module, then consider a map $f: Q_0 \to \mathbb{N}$, $q \mapsto \sum_{i=1}^{n} \dim \underline{\mathrm{Hom}}_\Lambda(M, \text{module at } (i,q) \text{ in } \mathbb{Z}Q/G)$ ← This is finite by the above

Claim

f is subadditive, but not additive.

Pf $\left\{ 2f(q) = \sum_{i=1}^{n} \dim \underline{\mathrm{Hom}}_A(M, \begin{smallmatrix}\text{module at}\\(i,q)\end{smallmatrix}) + \dim \underline{\mathrm{Hom}}_A(M, \begin{smallmatrix}\text{module at}\\(i-1,q)\end{smallmatrix}) \right.$

$\tau(\begin{smallmatrix}\text{module at}\\(i,q)\end{smallmatrix}) \;\; \Vert$

We have an AR-seq: $m=1$ if we assume $\mathbb{K}$ is alg. closed

$$0 \to \tau\begin{pmatrix}\text{mod at}\\(i,q)\end{pmatrix} \longrightarrow \bigoplus_{q\to q'}\begin{pmatrix}\text{mod at}\\(i-1,q')\end{pmatrix}^m \oplus \bigoplus_{q'\to q}\begin{pmatrix}\text{mod at}\\(i,q')\end{pmatrix}^m \to \begin{pmatrix}\text{mod at}\\(i,q)\end{pmatrix} \to 0$$

$$\Rightarrow \underline{\mathrm{Hom}}_A(M, \tau\begin{pmatrix}\text{mod at}\\(i,q)\end{pmatrix}) \longrightarrow \underline{\mathrm{Hom}}_A(M, \oplus \sim \oplus \oplus \sim) \to \underline{\mathrm{Hom}}_A(M, \begin{pmatrix}\text{mod at}\\(i,q)\end{pmatrix})$$

is exact, so
$$2f(q) \geq \sum_{i=1}^{n}\Big(\sum_{q\to q'} \dim \underline{\mathrm{Hom}}_A(M, \begin{pmatrix}\text{mod at}\\(i-1,q')\end{pmatrix}) + \sum_{q'\to q} \dim \underline{\mathrm{Hom}}_A(M, \begin{pmatrix}\text{mod at}\\(i,q')\end{pmatrix})\Big)$$
$$= \sum_{q\to q'} f(q') + \sum_{q'\to q} f(q')$$

So f is subadditive.

Let M be in our component of the AR-quiver. For the almost split seq. ending in M we have strict equality. $\square$

id_M will never be hit in the a.s. seq.

Hereditary algebras

Fact

$\mathbb{K}Q/I$ is hereditary if and only if $I = 0$.

admissible

Pf. Hereditary $\Leftrightarrow$ $\mathrm{rad}\, P_i$ is projective for any i.

$\mathrm{rad}\, P_i$ is the $\mathbb{K}$-lin. span of paths $\overbrace{\text{ending}}^{\text{non-trivial}}$ in i, up to relations.

proj. cov. of $\mathrm{rad}\, P_i$ is $\bigoplus_{j\to i} P_j = \mathbb{K}\{(\alpha,p) \mid \alpha : j \to i,\ p \text{ is a path ending in } j, \text{ up to } I\}$

$\circ : \bigoplus_{j\to i} P_j \to \mathrm{rad}\, P_i$, $(\alpha, p) \mapsto \alpha p$. This is iso iff $(\sum_\alpha \alpha p_\alpha = 0 \Rightarrow p_\alpha = 0 \ \forall \alpha)$

$\mathbb{K}$lin. comb. of paths

iff I has no $\overbrace{\text{relations}}^{\text{minimal}}$ ending in i. $\square$

Obs. A hereditary, then $\mathrm{Hom}_A(\begin{smallmatrix}\text{non-proj}\\\text{indec}\end{smallmatrix}, \text{proj}) = 0$ and $\mathrm{Hom}_A(\text{inj}, \begin{smallmatrix}\text{non-inj}\\\text{indec}\end{smallmatrix}) = 0$.

Assume that there is a morphism

$$\begin{tikzcd} \text{non-proj indec.} \arrow[rr] \arrow[dr, two heads] & & \text{proj} \\ & \mathrm{Im} \arrow[ur, hook] & \end{tikzcd}$$

Since A is hereditary, Im is proj and $\text{non-proj indec} \simeq \mathrm{Ker} \oplus \mathrm{Im} \Rightarrow \mathrm{Im} = 0$.

Prop

$A = \mathbb{K}Q$, Q connected

If A is representation finite, then the AR-quiver of A is a subspace of $\mathbb{N}Q$.

If A is representation infinite, then the AR-quiver has a component $\mathbb{N}Q$ containing the projectives ("preprojective comp."), and a component $(-\mathbb{N}Q)$ containing the injectives ("preinjective comp.").

Pf

AR-quiver restricted to projectives is Q. We start knitting:

- If we ever reach an injective, then we only have injectives after that, so we have a finite component and a finite AR-quiver $\Rightarrow$ rep-finite
- If we never reach an injective, then there is a component $\mathbb{N}Q$, likewise if we knit from injectives we get a component $(-\mathbb{N}Q)$. $\square$

Prop Let $A = \mathbb{K}Q$. If A is rep. finite, then Q is Dynkin.

Pf

sum is finite by rep. finite

Consider $f : i \mapsto \sum_{n=0}^{\infty} \dim \tau^{-n} P_i$

WTS: f is subadditive, not additive.

$$\begin{aligned} 2f(i) &= \dim P_i + (\dim P_i + \dim \tau^{-1} P_i) + \dots \\ &\quad + (\dim \tau^{-s+1} P_i + \dim \tau^{-s} P_i) + \dim \tau^{-s} P_i \\ &> \sum_{n=1}^{s} \dim \tau^{-n+1} P_i + \dim \tau^{-n} P_i + \dim \operatorname{rad} P_i + \dim \tau^{-s} P_i / \operatorname{soc} \tau^{-s} P_i \end{aligned}$$

$$= \dim \operatorname{rad} P_i + \sum_{n=0}^{s} \begin{pmatrix} \text{dim of middle} \\ \text{term of AR-seq} \\ \text{starting at } \tau^{-n} P_i \end{pmatrix} + \dim \tau^{-s} P_i / \operatorname{soc} \tau^{-s} P_i$$

$$= \sum_{\substack{i \to j \\ \text{or} \\ j \to i}} f(j) \qquad \square$$

Remark

The converse is true and may be checked by going through every Dynkin diagram.

Homological bilinear forms

Def The Grothendieck group of mod^A $\underbrace{}_{K_0(\text{mod}^A)}$ is the free abelian group on A-modules modulo $[M]=[L]+[N]$ where there are s.e. seq. $0 \to L \to M \to N \to 0$.

In particular $[L]+[N]=[L\oplus N]$

Fact $K_0(\text{mod}^A) \simeq \mathbb{Z}^n$, where $n = \#\text{simples}$ and it is given by the map $[M] \mapsto (\text{mult. of } S_i \text{ in a composition series of } M)_{i=1}^n$

and inverse $\sum_{i=1}^{n} v_i [S_i] \leftarrow\!\shortmid v$

Notation
$M \in \text{mod}^A$, we write $[M]$ for some el. in $K_0(A)$, and $\underline{\dim} M$ is the corresponding el. in $\mathbb{Z}^n$, called "dim. vector".

Rem If $A = \mathbb{K}Q/_I$, then $\underline{\dim}\, M = (\dim M_i)_{i=1}^n$.

Def Assume gl.dim is finite. Then the homological Euler form is given by $\langle M, N\rangle = \sum_{n=0}^{\infty} (-1)^n \dim \text{Ext}_A^n(M,N)$.

Fact $\langle -,-\rangle$ is a bilinear form on $K_0(\text{mod}^A)^2 \to \mathbb{Z}$.

The only issue for well-definedness is that this should only depend on the dimension vector.

$0 \to M_1 \to M_2 \to M_3 \to 0$ is short ex.

$$\Rightarrow \quad 0 \to \text{Hom}_A(M_3,N) \to \text{Hom}_A(M_2,N) \to \text{Hom}_A(M_1,N) \to \text{Ext}_A^1(M_3,N) \to \text{Ext}_A^1(M_2,N) \to \text{Ext}_A^1(M_1,N) \to \cdots$$

Then the alternating sum of dimensions are 0, i.e.

$$\dim(\text{Hom}(M_3,N)) - \dim(\text{Hom}(M_2,N)) + \dim(\text{Hom}(M_1,N)) - \cdots = 0$$

$\Rightarrow \langle M_3, N\rangle - \langle M_2, N\rangle + \langle M_1, N\rangle = 0$ ok.