Norwegian University of Science and Technology
Department of Mathematical Sciences

MA8404 Numerical solution of time dependent differential equations
Autumn 2017

Exercise set 4

1 Consider the problem with a Hamiltonian

$$H(p,q) = \frac{1}{2}p^2 + \frac{1}{2}q^2.$$

solved by the symplectic scheme:

$$q_{n+1} = q_n + hp_{n+1}, \qquad p_{n+1} = p_n - hq_n$$

a) Find the first few terms of the modified Hamiltonian.

b) Find the truncated modified equation:

$$q' = p + hf_2(p,q), \qquad p' = -q + hg_2(p,q).$$

Prove that the trajectories of of the solutions of this modified equation in the phase plane lie on one of the family of ellipses:

$$q(t)^2 - hq(t)p(t) + p(t)^2 = \text{constant}.$$

Solve the problem numerically, and verify that the numerical solution is close to this ellipse.

c) Find the the exact modified equation (expressed in terms of matrix functions.)

2 Given an ODE $y' = f(y)$, for which the nonlinear invariant $I(y)$ is conserved along the solution trajectories. Assume that the problem is solved by a method of order p, preserving the invariant. We will in this exercise prove that so will the corresponding modified equation

$$\tilde{y}' = f(\tilde{y}) + hg(\tilde{y}).$$

Remember: $I(y)$ is an invariant of f if $\nabla I(y) \cdot f(y) = 0$ for all y.

The proof is by contradiction: Assume that there is a point y^* for which the invariant is not satisfied, such that

$$\nabla I(y^*) \cdot g(y^*) > 0$$

(change the sign of I if necessary). Then there is a domain D around y^* such that

$$\nabla I(y) \cdot g(y) > 0, \qquad \forall y \in D$$

Choose $\tilde{y}(0) = y^*$ as the initial condition, and choose t_{end} such that $\tilde{y}(t)$ remain in D.

Show the following:

$$|I(\tilde{y}(t_{end}) - I(\tilde{y}(0))| > C \cdot t_{end} \cdot h^p$$

for some constant C (which one?), and

$$I(\tilde{y}(t_{end})) - I(\tilde{y}(0)) = \mathcal{O}(h^{p+1}).$$

These two can not simultanously be satisfied, so the modified equation will preserve the invariant.

NB! Sufficiently smoothness is assumed everywhere.

3 (Optional) Given the Kepler problem, with the Hamiltonian

$$H(p,q) = \frac{1}{2}p^T p - \frac{1}{r}, \qquad r = \|q\|_2 = \sqrt{q_1^2 + q_2^2}.$$

With initial values

$$q_0 = (1+\varepsilon, 0), \qquad p_0 = \left(0, \sqrt{\frac{1-\varepsilon}{1+\varepsilon}}\right)$$

the solution is an ellipse with eccentricity $\varepsilon \in [0, 1)$, with a period of 2π.

a) Solve this problem by Störmer-Verlets method over at least 10 periodes. Choose e.g. $h = 2\pi/50$, and try $\varepsilon = 0, 0.5, 0.9$. Experiment with different stepsizes.

Clearly, when the body is close to origin, the momentum p becomes large. It would therefore be advantage if the stepsize could be reduced there. As we saw in the lectures, using a standard stepsize controller will reduce the quality of the solution. The following strategy preserve symmetry, which can be proved to be sufficient to preserve qualitative behaviour of the solution.

b) Let Φ_h be the symplectic Euler method. Then implement the following strategy

$$\begin{aligned} y_{n+1/2} &= \Phi_{h/2}(y_n) \\ \frac{1}{h_{n+1}} + \frac{1}{h_n} &= \frac{2}{g(q_{n+1/2})}\frac{1}{h} \\ y_{n+1} &= \Phi^*_{h/2}(y_{n+1/2}). \end{aligned}$$

where $g(q) = 1/(1 + r^{-3/2})$, and h is some governing stepsize (constant).

Observe how the stepsize changes, see if the solution is improved, and plot also the Hamiltonian over time.

NB! For the method to preserve symmetry, we require g to be some smooth, positive, scalar-valued function which is invariant under $p \to -p$. So you may very well experiment with other functions as well.