Hidden Markov models

- Model definition:
  - unobserved state process, $\{x_t\}_{t=1}^n$: Markov chain

$$f(x_1, \ldots, x_n) = f(x_1) \cdot f(x_2|x_1) \cdot \ldots \cdot f(x_n|x_{n-1})$$

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- Distribution of interest

$$f(x_1, \ldots, x_n|y_1, \ldots, y_n) \quad \propto \quad f(x_1, \ldots, x_n) \cdot \prod_{i=1}^{n} f(y_i|x_i)$$

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$$\begin{aligned} f(x_1, \ldots, x_n|y_1, \ldots, y_n) &\propto f(x_1, \ldots, x_n) \cdot \prod_{i=1}^n f(y_i|x_i) \\ &= f(x_1)f(y_1|x_1) \cdot \prod_{i=2}^n [f(x_i|x_{i-1})f(y_i|x_i)] \end{aligned}$$

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  - note: we do not have the normalising constant for $f(x_1, \ldots, x_n|y_1, \ldots, y_n)$

Hidden Markov models

- Posterior distribution of interest

$$f(x_{1:n}|y_{1:n}) \propto g_{12}(x_1, x_2) \cdot g_{23}(x_2, x_3) \cdot \ldots \cdot g_{n-1,n}(x_{n-1}, x_n)$$

- We will see how to
  - find normalising constant (forward)
  - simulate (forward-backward)
  - find marginal distributions (forward-backward)
  - find state with maximum probability (Viterbi algorithm)

Normalising constant (when $n = 5$)

$$f(x_{1:5}|y_{1:5}) \quad = \quad c \cdot g_{12}(x_1, x_2) \cdot g_{23}(x_2, x_3) \cdot g_{34}(x_3, x_4) \cdot g_{45}(x_4, x_5)$$

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f(x_{2:5}|y_{1:5}) &= c \cdot \left[\sum_{x_1} g_{12}(x_1, x_2) \right] \cdot g_{23}(x_2, x_3) \cdot g_{34}(x_3, x_4) \cdot g_{45}(x_4, x_5)
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&= c \cdot h_2(x_2) \cdot g_{23}(x_2, x_3) \cdot g_{34}(x_3, x_4) \cdot g_{45}(x_4, x_5)
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&= c \cdot h_4(x_4) \cdot g_{45}(x_4, x_5) = c \cdot \widetilde{g}_{45}(x_4, x_5) \\
f(x_5|y_{1:5}) &= c \cdot \left[\sum_{x_4} \widetilde{g}_{45}(x_4, x_5) \right] = c \cdot h_5(x_5)
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&= c \cdot h_4(x_4) \cdot g_{45}(x_4, x_5) = c \cdot \widetilde{g}_{45}(x_4, x_5) \\
f(x_5|y_{1:5}) &= c \cdot \left[\sum_{x_4} \widetilde{g}_{45}(x_4, x_5) \right] = c \cdot h_5(x_5) \\
1 &= c \cdot \sum_{x_5} h_5(x_5)
\end{aligned}
$$

Normalising constant (when $n = 5$)

$$
\begin{aligned}
f(x_{1:5}|y_{1:5}) &= c \cdot g_{12}(x_1, x_2) \cdot g_{23}(x_2, x_3) \cdot g_{34}(x_3, x_4) \cdot g_{45}(x_4, x_5) \\
f(x_{2:5}|y_{1:5}) &= c \cdot \left[\sum_{x_1} g_{12}(x_1, x_2) \right] \cdot g_{23}(x_2, x_3) \cdot g_{34}(x_3, x_4) \cdot g_{45}(x_4, x_5) \\
&= c \cdot h_2(x_2) \cdot g_{23}(x_2, x_3) \cdot g_{34}(x_3, x_4) \cdot g_{45}(x_4, x_5) \\
&= c \cdot \widetilde{g}_{23}(x_2, x_3) \cdot g_{34}(x_3, x_4) \cdot g_{45}(x_4, x_5) \\
f(x_{3:5}|y_{1:5}) &= c \cdot \left[\sum_{x_2} \widetilde{g}_{23}(x_2, x_3) \right] \cdot g_{34}(x_3, x_4) \cdot g_{45}(x_4, x_5) \\
&= c \cdot h_3(x_3) \cdot g_{34}(x_3, x_4) \cdot g_{45}(x_4, x_5) \\
&= c \cdot \widetilde{g}_{34}(x_3, x_4) \cdot g_{45}(x_4, x_5) \\
f(x_{4:5}|y_{1:5}) &= c \cdot \left[\sum_{x_3} \widetilde{g}_{34}(x_3, x_4) \right] \cdot g_{45}(x_4, x_5) \\
&= c \cdot h_4(x_4) \cdot g_{45}(x_4, x_5) = c \cdot \widetilde{g}_{45}(x_4, x_5) \\
f(x_5|y_{1:5}) &= c \cdot \left[\sum_{x_4} \widetilde{g}_{45}(x_4, x_5) \right] = c \cdot h_5(x_5) \\
1 &= c \cdot \sum_{x_5} h_5(x_5) \Rightarrow c = 1 / \sum_{x_5} h_5(x_5)
\end{aligned}
$$