Let $y = x^*$. Assume we made a choice for $Q(y|x)$, what must $\alpha(y|x)$ be?

Recall the detailed balance condition:

$$\pi(x)\, P(y|x) = \pi(y)\, P(x|y) \qquad \forall x, y \in \Omega$$

We have

$$P(y|x) = \overset{\downarrow \text{proposal}}{Q(y|x)}\, \alpha(y|x) \qquad y \neq x$$

$$P(x|x) = 1 - \sum_{y \neq x} Q(y|x)\, \alpha(y|x)$$

Acceptance probability in the Metropolis-Hastings algorithm for $x \neq y$

$$\pi(x)\, Q(y|x)\, \alpha(y|x) = \underbrace{\pi(y)\, Q(x|y)}\, \alpha(x|y)$$

$$\alpha(y|x) = r(x,y)\, \pi(y)\, Q(x|y)$$

$$\Rightarrow \quad r(x,y) = \frac{\alpha(x|y)}{\pi(x)\, Q(y|x)}$$

$$\alpha(x|y) = r(y,x)\, \pi(x)\, Q(y|x)$$

$$\Rightarrow \quad r(y,x) = \frac{\alpha(y|x)}{\pi(y)\, Q(x|y)}$$

$$r(x,y) = r(y,x)$$

We must have that $\alpha(y|x) \in [0,1]$

$$\alpha(y|x) = r(x,y) Q(x|y) \pi(y) \leq 1$$

$$\Rightarrow r(x,y) \leq \frac{1}{Q(x|y)\pi(y)}$$

$$\alpha(x|y) = r(x,y) Q(y|x) \pi(x) \leq 1$$

$$\Rightarrow r(x,y) \leq \frac{1}{Q(y|x)\pi(x)}$$

$$\Rightarrow r(x,y) \leq \min\left\{\frac{1}{Q(x|y)\pi(y)}, \frac{1}{Q(y|x)\pi(x)}\right\}$$

choose $r(x,y)$ as large as possible

$$\Rightarrow r(x,y) = \min\left\{\frac{1}{Q(x|y)\pi(y)}, \frac{1}{Q(y|x)\pi(x)}\right\}$$

$$\alpha(y|x) = \min\left\{\frac{1}{Q(x|y)\pi(y)}, \frac{1}{Q(y|x)\pi(x)}\right\} Q(x|y)\pi(y)$$

$$= \min\left\{1, \frac{Q(x|y)\pi(y)}{Q(y|x)\pi(x)}\right\}$$

Lecture 2, Part 1. 19.01.2022

Discrete setting

$\pi(x)$ is a probability

$\{x_i\}_{i=0}^{\infty}$ is a MC with discrete state space

$Q(y|x)$ is a probability (transition matrix)

$\alpha(y|x)$ is a probability

$$P(y|x)=\begin{cases} Q(y|x)\,\alpha(y|x), & y\neq x \\ 1-\sum\limits_{z\neq x} Q(z|x)\,\alpha(z|x), & y=x \end{cases}$$

is a probability (transition matrix)

Limiting distribution

$$\pi(y)=\sum_x \pi(x)\,P(y|x)$$

Detailed balance condition

$$\pi(x)\,P(y|x)=\pi(y)\,P(x|y) \quad \forall x,y$$

Continuous setting

$\pi(x)$ is a density

$\{x_i\}_{i=0}^{\infty}$ is a MC with continuous state space

$Q(y|x)$ is a density

$$\int Q(y|x)\,dy = 1$$

$\alpha(y|x)$ is a probability

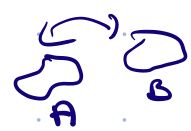
$$P(A|x)=\int_A Q(y|x)\,\alpha(y|x)\,dy + \overset{\text{indicator}}{I}(x\in A)\int Q(y|x)\,(1-\alpha(y|x))\,dy$$

is a transition kernel

Limiting distribution

$$\int_A \pi(x)\,dx=\int_\Omega \pi(x)\,P(A|x)\,dx$$

$$\int_B \pi(x)\,P(A|x)\,dx=\int_A \pi(y)\,P(B|y)\,dy$$



Comment:

$$P(x|x)=Q(x|x)\overbrace{\alpha(x|x)}^{1}+\sum_{z\neq x} Q(z|x)\,(1-\alpha(z|x))$$

$$=Q(x|x)+\sum_{z\neq x} Q(z|x)-\sum_{z\neq x} Q(z|x)\,\alpha(z|x)$$

$$=1-\sum_{z\neq x} Q(z|x)\,\alpha(z|x)$$

Setting $P(A|x) = \int_A Q(y|x)\alpha(y|x)\,dy +$

$$I(x \in A) \int Q(y|x)(1-\alpha(y|x))\,dy$$

into the detailled balance condition you get something corresponding to the discrete version

$$\alpha(y|x) = \min\left\{1, \frac{\pi(y)}{\pi(x)} \cdot \frac{Q(x|y)}{Q(y|x)}\right\}$$