1. Let $X_1, X_2, ...$ be a sequence of random variables. Each X_n takes two values, $\sqrt{n}$ and $(n+1)/n$, with probabilities

$$P(X_n = (n+1)/n) = 1 - \frac{1}{n}, \quad P(X_n = \sqrt{n}) = \frac{1}{n}.$$

Does the sequence $\{X_n\}$ converge

a) in probability?

b) in L^1?

c) in L^2?

2. Let $X_1, X_2, ...$ be iid random variables with the density

$$f(x) = 2(1-x)I_{[0,1]}(x).$$

Define

$$Y_n = \prod_{k=1}^{n} X_k.$$

Prove that the series

$$\sum_{n=1}^{\infty} Y_n$$

a.s. converges.

3. $X_1, X_2, ...$ are independent random variables, and

$$X_n \xrightarrow{\text{a.s.}} X, \ n \to \infty.$$

Denote the characteristic function of X by $\varphi_X(t)$. Prove that $|\varphi_X(t)| \equiv 1$.

4. Give an example of *dependent* random variables $X_1, X_2, ...$ such that $\{X_n\}$ satisfies the central limit theorem, i.e.

$$\frac{S_n - ES_n}{\sqrt{VarS_n}} \xrightarrow{\text{d}} N(0,1), \quad n \to \infty,$$

where $S_n = X_1 + ... + X_n$.