1. Let $A_1, A_2, ...$ be a sequence of events. Is it correct that

$$\liminf_{n\to\infty} P(A_n) > 0$$

implies

$$P(\liminf_n A_n) > 0?$$

Prove or give a counterexample.

2. Let X and Y be independent random variables with continuous distribution functions. Prove that the distribution function of the sum $X+Y$ is also continuous. Moreover, prove that

$$\sup_x P(x < X+Y < x+\Delta) \le$$

$$\le \min\{\sup_x P(x < X < x+\Delta), \sup_x P(x < Y < x+\Delta)\}$$

for any $\Delta > 0$.

3. Let $X_1, X_2, ...$ be a sequence of random variables. Each X_n takes two values, $e^{1/n}$ and e^n, with probabilities

$$P(X_n = e^{1/n}) = 1 - \frac{1}{n}, \quad P(X_n = e^n) = \frac{1}{n}.$$

Does the sequence $\{X_n\}$ converge

a) in distribution?

a) in probability?

b) in L^1?

4. $X_1, X_2, ...$ are independent random variables, and

$$X_n \xrightarrow{\text{P}} X, \ n \to \infty.$$

Find $Var(X)$.

5. Are the following functions characteristic functions?

a)

$$\cos\frac{1}{t}$$

b)

$$\sin t$$

c)

$$e^{2021(e^{it}-1)}$$

6. Let X and Y be iid random variables. Prove that for any $\varepsilon > 0$ and any real c

$$P(|X-Y| \ge \varepsilon) \le 2P(|X-c| \ge \varepsilon/2).$$