A third option is to state a mathematical formula that the sample outcomes must satisfy.

A computer programmer is running a subroutine that solves a general quadratic equation, $ax^2 + bx + c = 0$. Her "experiment" consists of choosing values for the three coefficients a, b, and c. Define (1) S and (2) the event A: Equation has two equal roots.

First, we must determine the sample space. Since presumably no combinations of finite a, b, and c are inadmissible, we can characterize S by writing a series of inequalities:

$$S = \{(a, b, c) : -\infty < a < \infty, -\infty < b < \infty, -\infty < c < \infty\}$$

Defining A requires the well-known result from algebra that a quadratic equation has equal roots if and only if its discriminant, $b^2 - 4ac$, vanishes. Membership in A, then, is contingent on a, b, and c satisfying an equation:

$$A = \{(a, b, c) : b^2 - 4ac = 0\}$$ ■

Questions

2.2.1. A graduating engineer has signed up for three job interviews. She intends to categorize each one as being either a "success" or a "failure" depending on whether it leads to a plant trip. Write out the appropriate sample space. What outcomes are in the event A: Second success occurs on third interview? In B: First success never occurs? (*Hint*: Notice the similarity between this situation and the coin-tossing experiment described in Example 2.2.1.)

2.2.2. Three dice are tossed, one red, one blue, and one green. What outcomes make up the event A that the sum of the three faces showing equals 5?

2.2.3. An urn contains six chips numbered 1 through 6. Three are drawn out. What outcomes are in the event "Second smallest chip is a 3"? Assume that the order of the chips is irrelevant.

2.2.4. Suppose that two cards are dealt from a standard 52-card poker deck. Let A be the event that the sum of the two cards is 8 (assume that aces have a numerical value of 1). How many outcomes are in A?

2.2.5. In the lingo of craps-shooters (where two dice are tossed and the underlying sample space is the matrix pictured in Figure 2.2.1) is the phrase "making a hard eight." What might that mean?

2.2.6. A poker deck consists of fifty-two cards, representing thirteen denominations (2 through ace) and four suits (diamonds, hearts, clubs, and spades). A five-card hand is called a *flush* if all five cards are in the same suit but not all five denominations are consecutive. Pictured in the next column is a flush in hearts. Let N be the set of five cards in hearts that are *not* flushes. How many outcomes are in N? [*Note*: In poker, the denominations (A, 2, 3, 4, 5) are considered to be consecutive (in addition to sequences such as (8, 9, 10, J, Q)).]

		Denominations												
		2	3	4	5	6	7	8	9	10	J	Q	K	A
Suits	D													
	H	X	X				X				X	X		
	C													
	S													

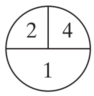
2.2.7. Let P be the set of right triangles with a 5″ hypotenuse and whose height and length are a and b, respectively. Characterize the outcomes in P.

2.2.8. Suppose a baseball player steps to the plate with the intention of trying to "coax" a base on balls by never swinging at a pitch. The umpire, of course, will necessarily call each pitch either a ball (B) or a strike (S). What outcomes make up the event A, that a batter walks on the sixth pitch? (*Note*: A batter "walks" if the fourth ball is called before the third strike.)

2.2.9. A telemarketer is planning to set up a phone bank to bilk widows with a Ponzi scheme. His past experience (prior to his most recent incarceration) suggests that each phone will be in use half the time. For a given phone at a given time, let 0 indicate that the phone is available and let 1 indicate that a caller is on the line. Suppose that the telemarketer's "bank" is comprised of four telephones.

(a) Write out the outcomes in the sample space.
(b) What outcomes would make up the event that exactly two phones are being used?
(c) Suppose the telemarketer had k phones. How many outcomes would allow for the possibility that at most one more call could be received? (*Hint*: How many lines would have to be busy?)

2.2.10. Two darts are thrown at the following target:



(a) Let (u, v) denote the outcome that the first dart lands in region u and the second dart, in region v. List the sample space of (u, v)'s.
(b) List the outcomes in the sample space of *sums*, $u + v$.

2.2.11. A woman has her purse snatched by two teenagers. She is subsequently shown a police lineup consisting of five suspects, including the two perpetrators. What is the sample space associated with the experiment "Woman picks two suspects out of lineup"? Which outcomes are in the event A: She makes at least one incorrect identification?

2.2.12. Consider the experiment of choosing coefficients for the quadratic equation $ax^2 + bx + c = 0$. Characterize the values of a, b, and c associated with the event A: Equation has complex roots.

2.2.13. In the game of craps, the person rolling the dice (the *shooter*) wins outright if his first toss is a 7 or an 11. If his first toss is a 2, 3, or 12, he loses outright. If his first roll is something else, say, a 9, that number becomes his "point" and he keeps rolling the dice until he either rolls another 9, in which case he wins, or a 7, in which case he loses. Characterize the sample outcomes contained in the event "Shooter wins with a point of 9."

2.2.14. A probability-minded despot offers a convicted murderer a final chance to gain his release. The prisoner is given twenty chips, ten white and ten black. All twenty are to be placed into two urns, according to any allocation scheme the prisoner wishes, with the one proviso being that each urn contain at least one chip. The executioner will then pick one of the two urns at random and from that urn, one chip at random. If the chip selected is white, the prisoner will be set free; if it is black, he "buys the farm." Characterize the sample space describing the prisoner's possible allocation options. (Intuitively, which allocation affords the prisoner the greatest chance of survival?)

2.2.15. Suppose that ten chips, numbered 1 through 10, are put into an urn at one minute to midnight, and chip number 1 is quickly removed. At one-half minute to midnight, chips numbered 11 through 20 are added to the urn, and chip number 2 is quickly removed. Then at one-fourth minute to midnight, chips numbered 21 to 30 are added to the urn, and chip number 3 is quickly removed. If that procedure for adding chips to the urn continues, how many chips will be in the urn at midnight (148)?

Unions, Intersections, and Complements

Associated with events defined on a sample space are several operations collectively referred to as the *algebra of sets*. These are the rules that govern the ways in which one event can be combined with another. Consider, for example, the game of craps described in Question 2.2.13. The shooter wins on his initial roll if he throws either a 7 or an 11. In the language of the algebra of sets, the event "Shooter rolls a 7 *or* an 11" is the *union* of two simpler events, "Shooter rolls a 7" and "Shooter rolls an 11." If E denotes the union and if A and B denote the two events making up the union, we write $E = A \cup B$. The next several definitions and examples illustrate those portions of the algebra of sets that we will find particularly useful in the chapters ahead.

Definition 2.2.1. Let A and B be any two events defined over the same sample space S. Then

a. The *intersection* of A and B, written $A \cap B$, is the event whose outcomes belong to both A and B.
b. The *union* of A and B, written $A \cup B$, is the event whose outcomes belong to either A or B or both.

Example 2.2.9 Consider a single throw of two dice. Define A to be the event that the *sum* of the faces showing is odd. Let B be the event that the two faces themselves are odd. Then clearly, the intersection is empty, the sum of two odd numbers necessarily being even. In symbols, $A \cap B = \emptyset$. (Recall the event $B \cap C$ asked for in Example 2.2.6.) ■

> **Definition 2.2.3.** Let A be any event defined on a sample space S. The *complement* of A, written A^C, is the event consisting of all the outcomes in S other than those contained in A.

Example 2.2.10 Let A be the set of (x, y)'s for which $x^2 + y^2 < 1$. Sketch the region in the xy-plane corresponding to A^C.

From analytic geometry, we recognize that $x^2 + y^2 < 1$ describes the interior of a circle of radius 1 centered at the origin. Figure 2.2.3 shows the complement—the points on the circumference of the circle and the points outside the circle.

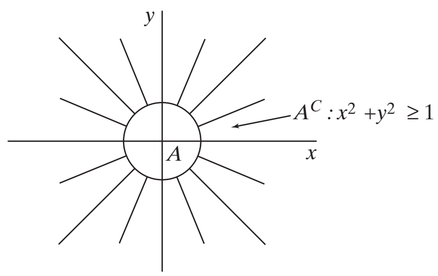


Figure 2.2.3 ■

The notions of union and intersection can easily be extended to more than two events. For example, the expression $A_1 \cup A_2 \cup \cdots \cup A_k$ defines the set of outcomes belonging to *any* of the A_i's (or to any combination of the A_i's). Similarly, $A_1 \cap A_2 \cap \cdots \cap A_k$ is the set of outcomes belonging to *all* of the A_i's.

Example 2.2.11 Suppose the events $A_1, A_2, \ldots, A_k$ are intervals of real numbers such that

$$A_i = \{x : 0 \leq x < 1/i\}, \quad i = 1, 2, \ldots, k$$

Describe the sets $A_1 \cup A_2 \cup \cdots \cup A_k = \cup_{i=1}^{k} A_i$ and $A_1 \cap A_2 \cap \cdots \cap A_k = \cap_{i=1}^{k} A_i$.

Notice that the A_i's are telescoping sets. That is, A_1 is the interval $0 \leq x < 1$, A_2 is the interval $0 \leq x < \frac{1}{2}$, and so on. It follows, then, that the *union* of the k A_i's is simply A_1 while the *intersection* of the A_i's (that is, their overlap) is A_k. ■

Questions

2.2.16. Sketch the regions in the xy-plane corresponding to $A \cup B$ and $A \cap B$ if

$$A = \{(x, y): 0 < x < 3, 0 < y < 3\}$$

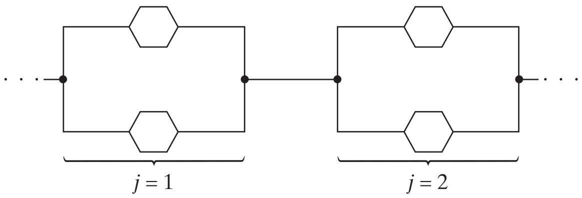
and

$$B = \{(x, y): 2 < x < 4, 2 < y < 4\}$$

2.2.17. Referring to Example 2.2.7, find $A \cap B$ and $A \cup B$ if the two equations were replaced by inequalities: $x^2 + 2x \leq 8$ and $x^2 + x \leq 6$.

2.2.18. Find $A \cap B \cap C$ if $A = \{x: 0 \leq x \leq 4\}$, $B = \{x: 2 \leq x \leq 6\}$, and $C = \{x: x = 0, 1, 2, \ldots\}$.

2.2.19. An electronic system has four components divided into two pairs. The two components of each pair are wired in parallel; the two pairs are wired in series. Let A_{ij} denote the event "ith component in jth pair fails," $i = 1, 2$; $j = 1, 2$. Let A be the event "System fails." Write A in terms of the A_{ij}'s.



2.2.20. Define $A = \{x : 0 \le x \le 1\}$, $B = \{x : 0 \le x \le 3\}$, and $C = \{x : -1 \le x \le 2\}$. Draw diagrams showing each of the following sets of points:

(a) $A^C \cap B \cap C$
(b) $A^C \cup (B \cap C)$
(c) $A \cap B \cap C^C$
(d) $[(A \cup B) \cap C^C]^C$

2.2.21. Let A be the set of five-card hands dealt from a 52-card poker deck, where the denominations of the five cards are all consecutive—for example, (7 of hearts, 8 of spades, 9 of spades, 10 of hearts, jack of diamonds). Let B be the set of five-card hands where the suits of the five cards are all the same. How many outcomes are in the event $A \cap B$?

2.2.22. Suppose that each of the twelve letters in the word

$$T \quad E \quad S \quad S \quad E \quad L \quad L \quad A \quad T \quad I \quad O \quad N$$

is written on a chip. Define the events $F, R,$ and C as follows:

F: letters in first half of alphabet
R: letters that are repeated
V: letters that are vowels

Which chips make up the following events?

(a) $F \cap R \cap V$
(b) $F^C \cap R \cap V^C$
(c) $F \cap R^C \cap V$

2.2.23. Let A, B, and C be any three events defined on a sample space S. Show that

(a) the outcomes in $A \cup (B \cap C)$ are the same as the outcomes in $(A \cup B) \cap (A \cup C)$.
(b) the outcomes in $A \cap (B \cup C)$ are the same as the outcomes in $(A \cap B) \cup (A \cap C)$.

2.2.24. Let $A_1, A_2, \ldots, A_k$ be any set of events defined on a sample space S. What outcomes belong to the event

$$(A_1 \cup A_2 \cup \cdots \cup A_k) \cup \left(A_1^C \cap A_2^C \cap \cdots \cap A_k^C\right)$$

2.2.25. Let $A, B,$ and C be any three events defined on a sample space S. Show that the operations of union and intersection are *associative* by proving that

(a) $A \cup (B \cup C) = (A \cup B) \cup C = A \cup B \cup C$
(b) $A \cap (B \cap C) = (A \cap B) \cap C = A \cap B \cap C$

2.2.26. Suppose that three events—$A, B,$ and C—are defined on a sample space S. Use the union, intersection, and complement operations to represent each of the following events:

(a) none of the three events occurs
(b) all three of the events occur
(c) only event A occurs
(d) exactly one event occurs
(e) exactly two events occur

2.2.27. What must be true of events A and B if

(a) $A \cup B = B$
(b) $A \cap B = A$

2.2.28. Let events A and B and sample space S be defined as the following intervals:

$$S = \{x : 0 \le x \le 10\}$$
$$A = \{x : 0 < x < 5\}$$
$$B = \{x : 3 \le x \le 7\}$$

Characterize the following events:

(a) A^C
(b) $A \cap B$
(c) $A \cup B$
(d) $A \cap B^C$
(e) $A^C \cup B$
(f) $A^C \cap B^C$

2.2.29. A coin is tossed four times and the resulting sequence of heads and/or tails is recorded. Define the events A, B, and C as follows:

A: exactly two heads appear
B: heads and tails alternate
C: first two tosses are heads

(a) Which events, if any, are mutually exclusive?
(b) Which events, if any, are subsets of other sets?

2.2.30. Pictured on the next page are two organizational charts describing the way upper management vets new proposals. For both models, three vice presidents—1, 2, and 3—each voice an opinion.

b. The event "Neither appears" is the complement of the event "At least one appears." But $P(\text{At least one appears}) = P(L \cup D)$. From Theorems 2.3.1 and 2.3.6, then,

$$\begin{aligned} P(\text{Neither appears}) &= 1 - P(L \cup D) \\ &= 1 - [P(L) + P(D) - P(L \cap D)] \\ &= 1 - [0.40 + 0.30 - 0.05] \\ &= 0.35 \end{aligned}$$

Example 2.3.5 Having endured (and survived) the mental trauma that comes from taking two years of chemistry, a year of physics, and a year of biology, Biff decides to test the medical school waters and sends his MCATs to two colleges, X and Y. Based on how his friends have fared, he estimates that his probability of being accepted at X is 0.7, and at Y is 0.4. He also suspects there is a 75% chance that at least one of his applications will be rejected. What is the probability that he gets at least one acceptance?

Let A be the event "School X accepts him" and B the event "School Y accepts him." We are given that $P(A) = 0.7$, $P(B) = 0.4$, and $P(A^C \cup B^C) = 0.75$. The question is asking for $P(A \cup B)$.

From Theorem 2.3.6,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Recall from Question 2.2.32 that $A^C \cup B^C = (A \cap B)^C$, so

$$P(A \cap B) = 1 - P[(A \cap B)^C] = 1 - 0.75 = 0.25$$

It follows that Biff's prospects are not all that bleak—he has an 85% chance of getting in somewhere:

$$\begin{aligned} P(A \cup B) &= 0.7 + 0.4 - 0.25 \\ &= 0.85 \end{aligned}$$

Comment Notice that $P(A \cup B)$ varies directly with $P(A^C \cup B^C)$:

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - [1 - P(A^C \cup B^C)] \\ &= P(A) + P(B) - 1 + P(A^C \cup B^C) \end{aligned}$$

If $P(A)$ and $P(B)$, then, are fixed, we get the curious result that Biff's chances of getting at least one acceptance increase if his chances of at least one rejection increase.

Questions

2.3.1. According to a family-oriented lobbying group, there is too much crude language and violence on television. Forty-two percent of the programs they screened had language they found offensive, 27% were too violent, and 10% were considered excessive in both language and violence. What percentage of programs did comply with the group's standards?

2.3.2. Let A and B be any two events defined on S. Suppose that $P(A) = 0.4$, $P(B) = 0.5$, and $P(A \cap B) = 0.1$. What is the probability that A or B but not both occur?

2.3.3. Express the following probabilities in terms of $P(A)$, $P(B)$, and $P(A \cap B)$.

(a) $P(A^C \cup B^C)$
(b) $P(A^C \cap (A \cup B))$

2.3.4. Let A and B be two events defined on S. If the probability that at least one of them occurs is 0.3 and the probability that A occurs but B does not occur is 0.1, what is $P(B)$?

2.3.5. Suppose that three fair dice are tossed. Let A_i be the event that a 6 shows on the ith die, $i = 1, 2, 3$. Does $P(A_1 \cup A_2 \cup A_3) = \frac{1}{2}$? Explain.

2.3.6. Events A and B are defined on a sample space S such that $P((A \cup B)^C) = 0.5$ and $P(A \cap B) = 0.2$. What is the probability that either A or B but not both will occur?

2.3.7. Let $A_1, A_2, \ldots, A_n$ be a series of events for which $A_i \cap A_j = \emptyset$ if $i \neq j$ and $A_1 \cup A_2 \cup \cdots \cup A_n = S$. Let B be any event defined on S. Express B as a union of intersections.

2.3.8. Draw the Venn diagrams that would correspond to the equations (a) $P(A \cap B) = P(B)$ and (b) $P(A \cup B) = P(B)$.

2.3.9. In the game of "odd man out" each player tosses a fair coin. If all the coins turn up the same except for one, the player tossing the different coin is declared the odd man out and is eliminated from the contest. Suppose that three people are playing. What is the probability that someone will be eliminated on the first toss? (*Hint:* Use Theorem 2.3.1.)

2.3.10. An urn contains twenty-four chips, numbered 1 through 24. One is drawn at random. Let A be the event that the number is divisible by 2 and let B be the event that the number is divisible by 3. Find $P(A \cup B)$.

2.3.11. If State's football team has a 10% chance of winning Saturday's game, a 30% chance of winning two weeks from now, and a 65% chance of losing both games, what are their chances of winning exactly once?

2.3.12. Events A_1 and A_2 are such that $A_1 \cup A_2 = S$ and $A_1 \cap A_2 = \emptyset$. Find p_2 if $P(A_1) = p_1$, $P(A_2) = p_2$, and $3p_1 - p_2 = \frac{1}{2}$.

2.3.13. Consolidated Industries has come under considerable pressure to eliminate its seemingly discriminatory hiring practices. Company officials have agreed that during the next five years, 60% of their new employees will be females and 30% will be minorities. One out of four new employees, though, will be a white male. What percentage of their new hires will be minority females?

2.3.14. Three events—A, B, and C—are defined on a sample space, S. Given that $P(A) = 0.2$, $P(B) = 0.1$, and $P(C) = 0.3$, what is the smallest possible value for $P[(A \cup B \cup C)^C]$?

2.3.15. A coin is to be tossed four times. Define events X and Y such that

X: first and last coins have opposite faces
Y: exactly two heads appear

Assume that each of the sixteen head/tail sequences has the same probability. Evaluate

(a) $P(X^C \cap Y)$
(b) $P(X \cap Y^C)$

2.3.16. Two dice are tossed. Assume that each possible outcome has a $\frac{1}{36}$ probability. Let A be the event that the sum of the faces showing is 6, and let B be the event that the face showing on one die is twice the face showing on the other. Calculate $P(A \cap B^C)$.

2.3.17. Let A, B, and C be three events defined on a sample space, S. Arrange the probabilities of the following events from smallest to largest:

(a) $A \cup B$
(b) $A \cap B$
(c) A
(d) S
(e) $(A \cap B) \cup (A \cap C)$

2.3.18. Lucy is currently running two dot-com scams out of a bogus chatroom. She estimates that the chances of the first one leading to her arrest are one in ten; the "risk" associated with the second is more on the order of one in thirty. She considers the likelihood that she gets busted for both to be 0.0025. What are Lucy's chances of avoiding incarceration?

2.4 Conditional Probability

In Section 2.3, we calculated probabilities of certain events by manipulating other probabilities whose values we were given. Knowing $P(A)$, $P(B)$, and $P(A \cap B)$, for example, allows us to calculate $P(A \cup B)$ (recall Theorem 2.3.6). For many real-world situations, though, the "given" in a probability problem goes beyond simply knowing a set of other probabilities. Sometimes, we know *for a fact* that certain events *have already occurred*, and those occurrences may have a bearing on the