Let A be the event that the prize is behind Door #2, and let B be the event that the host opened Door #3. Then

$$P(A|B) = P(\text{Contestant wins by not switching}) = [P(A \cap B)]/P(B)$$
$$= \left[\tfrac{1}{6}\right] / \left[\tfrac{1}{3} + \tfrac{1}{6}\right]$$
$$= \tfrac{1}{3}$$

Now, let A^* be the event that the prize is behind Door #1, and let B (as before) be the event that the host opens Door #3. In this case,

$$P(A^*|B) = P(\text{Contestant wins by switching}) = [P(A^* \cap B)]/P(B)$$
$$= \left[\tfrac{1}{3}\right] / \left[\tfrac{1}{3} + \tfrac{1}{6}\right]$$
$$= \tfrac{2}{3}$$

Common sense would have led us astray again! If given the choice, contestants should have *always* switched doors. Doing so upped their chances of winning from one-third to two-thirds.

Questions

2.4.1. Suppose that two fair dice are tossed. What is the probability that the sum equals 10 given that it exceeds 8?

2.4.2. Find $P(A \cap B)$ if $P(A) = 0.2$, $P(B) = 0.4$, and $P(A|B) + P(B|A) = 0.75$.

2.4.3. If $P(A|B) < P(A)$, show that $P(B|A) < P(B)$.

2.4.4. Let A and B be two events such that $P((A \cup B)^C) = 0.6$ and $P(A \cap B) = 0.1$. Let E be the event that either A or B but not both will occur. Find $P(E|A \cup B)$.

2.4.5. Suppose that in Example 2.4.2 we ignored the ages of the children and distinguished only *three* family types: (boy, boy), (girl, boy), and (girl, girl). Would the conditional probability of both children being boys given that at least one is a boy be different from the answer found on p. 35? Explain.

2.4.6. Two events, A and B, are defined on a sample space S such that $P(A|B) = 0.6$, P(At least one of the events occurs) $= 0.8$, and P(Exactly one of the events occurs) $= 0.6$. Find $P(A)$ and $P(B)$.

2.4.7. An urn contains one red chip and one white chip. One chip is drawn at random. If the chip selected is red, that chip together with two additional red chips are put back into the urn. If a white chip is drawn, the chip is returned to the urn. Then a second chip is drawn. What is the probability that both selections are red?

2.4.8. Given that $P(A) = a$ and $P(B) = b$, show that

$$P(A|B) \geq \frac{a + b - 1}{b}$$

2.4.9. An urn contains one white chip and a second chip that is equally likely to be white or black. A chip is drawn at random and returned to the urn. Then a second chip is drawn. What is the probability that a white appears on the second draw given that a white appeared on the first draw? (*Hint:* Let W_i be the event that a white chip is selected on the ith draw, $i = 1, 2$. Then $P(W_2|W_1) = \frac{P(W_1 \cap W_2)}{P(W_1)}$. If both chips in the urn are white, $P(W_1) = 1$; otherwise, $P(W_1) = \frac{1}{2}$.)

2.4.10. Suppose events A and B are such that $P(A \cap B) = 0.1$ and $P((A \cup B)^C) = 0.3$. If $P(A) = 0.2$, what does $P[(A \cap B)|(A \cup B)^C]$ equal? (*Hint*: Draw the Venn diagram.)

2.4.11. One hundred voters were asked their opinions of two candidates, A and B, running for mayor. Their responses to three questions are summarized below:

	Number Saying "Yes"
Do you like A?	65
Do you like B?	55
Do you like both?	25

(a) What is the probability that someone likes neither?
(b) What is the probability that someone likes exactly one?
(c) What is the probability that someone likes at least one?
(d) What is the probability that someone likes at most one?

(e) What is the probability that someone likes exactly one given that he or she likes at least one?
(f) Of those who like at least one, what proportion like both?
(g) Of those who do not like A, what proportion like B?

2.4.12. A fair coin is tossed three times. What is the probability that at least two heads will occur given that at most two heads have occurred?

2.4.13. Two fair dice are rolled. What is the probability that the number on the first die was at least as large as 4 given that the sum of the two dice was 8?

2.4.14. Four cards are dealt from a standard 52-card poker deck. What is the probability that all four are aces given that at least three are aces? (*Note*: There are 270,725 different sets of four cards that can be dealt. Assume that the probability associated with each of those hands is 1/270,725.)

2.4.15. Given that $P(A \cap B^C) = 0.3$, $P((A \cup B)^C) = 0.2$, and $P(A \cap B) = 0.1$, find $P(A|B)$.

2.4.16. Given that $P(A) + P(B) = 0.9$, $P(A|B) = 0.5$, and $P(B|A) = 0.4$, find $P(A)$.

2.4.17. Let A and B be two events defined on a sample space S such that $P(A \cap B^C) = 0.1$, $P(A^C \cap B) = 0.3$, and $P((A \cup B)^C) = 0.2$. Find the probability that at least one of the two events occurs given that at most one occurs.

2.4.18. Suppose two dice are rolled. Assume that each possible outcome has probability 1/36. Let A be the event that the sum of the two dice is greater than or equal to 8, and let B be the event that at least one of the dice shows a 5. Find $P(A|B)$.

2.4.19. According to your neighborhood bookie, five horses are scheduled to run in the third race at the local track, and handicappers have assigned them the following probabilities of winning:

Horse	Probability of Winning
Scorpion	0.10
Starry Avenger	0.25
Australian Doll	0.15
Dusty Stake	0.30
Outandout	0.20

Suppose that Australian Doll and Dusty Stake are scratched from the race at the last minute. What are the chances that Outandout will prevail over the reduced field?

2.4.20. Andy, Bob, and Charley have all been serving time for grand theft auto. According to prison scuttlebutt, the warden plans to release two of the three next week. They all have identical records, so the two to be released will be chosen at random, meaning that each has a two-thirds probability of being included in the two to be set free. Andy, however, is friends with a guard who will know ahead of time which two will leave. He offers to tell Andy the name of one prisoner *other than himself* who will be released. Andy, however, declines the offer, believing that if he learns the name of one prisoner scheduled to be released, then *his* chances of being the other person set free will drop to one-half (since only two prisoners will be left at that point). Is his concern justified?

Applying Conditional Probability to Higher-Order Intersections

We have seen that conditional probabilities can be useful in evaluating intersection probabilities—that is, $P(A \cap B) = P(A|B)P(B) = P(B|A)P(A)$. A similar result holds for higher-order intersections. Consider $P(A \cap B \cap C)$. By thinking of $A \cap B$ as a single event—say, D—we can write

$$\begin{aligned} P(A \cap B \cap C) &= P(D \cap C) \\ &= P(C|D)P(D) \\ &= P(C|A \cap B)P(A \cap B) \\ &= P(C|A \cap B)P(B|A)P(A) \end{aligned}$$

Repeating this same argument for n events, $A_1, A_2, \ldots, A_n$, gives a formula for the general case:

$$\begin{aligned} P(A_1 \cap A_2 \cap \cdots \cap A_n) = {} & P(A_n|A_1 \cap A_2 \cap \cdots \cap A_{n-1}) \\ & \cdot P(A_{n-1}|A_1 \cap A_2 \cap \cdots \cap A_{n-2}) \cdot \cdots \cdot P(A_2|A_1) \cdot P(A_1) \end{aligned} \tag{2.4.3}$$

3.0. Based on those estimates, what is the probability that Marcus gets into medical school?

2.4.38. The governor of a certain state has decided to come out strongly for prison reform and is preparing a new early release program. Its guidelines are simple: prisoners related to members of the governor's staff would have a 90% chance of being released early; the probability of early release for inmates not related to the governor's staff would be 0.01. Suppose that 40% of all inmates are related to someone on the governor's staff. What is the probability that a prisoner selected at random would be eligible for early release?

2.4.39. Following are the percentages of students of State College enrolled in each of the school's main divisions. Also listed are the proportions of students in each division who are women.

Division	%	% Women
Humanities	40	60
Natural science	10	15
History	30	45
Social science	20	75
	100	

Suppose the registrar selects one person at random. What is the probability that the student selected will be a male?

Bayes' Theorem

The second result in this section that is set against the backdrop of a partitioned sample space has a curious history. The first explicit statement of Theorem 2.4.2, coming in 1812, was due to Laplace, but it was named after the Reverend Thomas Bayes, whose 1763 paper (published posthumously) had already outlined the result. On one level, the theorem is a relatively minor extension of the definition of conditional probability. When viewed from a loftier perspective, though, it takes on some rather profound philosophical implications. The latter, in fact, have precipitated a schism among practicing statisticians: "Bayesians" analyze data one way; "non-Bayesians" often take a fundamentally different approach (see Section 5.8).

Our use of the result here will have nothing to do with its statistical interpretation. We will apply it simply as the Reverend Bayes originally intended, as a formula for evaluating a certain kind of "inverse" probability. If we know $P(B|A_i)$ for all i, the theorem enables us to compute conditional probabilities "in the other direction"—that is, we can deduce $P(A_j|B)$ from the $P(B|A_i)$'s.

Theorem 2.4.2 *(Bayes') Let $\{A_i\}_{i=1}^n$ be a set of n events, each with positive probability, that partition S in such a way that $\cup_{i=1}^n A_i = S$ and $A_i \cap A_j = \emptyset$ for $i \neq j$. For any event B (also defined on S), where $P(B) > 0$,*

$$P(A_j|B) = \frac{P(B|A_j)P(A_j)}{\sum_{i=1}^{n} P(B|A_i)P(A_i)}$$

for any $1 \leq j \leq n$.

Proof From Definition 2.4.1,

$$P(A_j|B) = \frac{P(A_j \cap B)}{P(B)} = \frac{P(B|A_j)P(A_j)}{P(B)}$$

But Theorem 2.4.1 allows the denominator to be written as $\sum_{i=1}^{n} P(B|A_i)P(A_i)$, and the result follows. □

2.4.46. Brett and Margo have each thought about murdering their rich Uncle Basil in hopes of claiming their inheritance a bit early. Hoping to take advantage of Basil's predilection for immoderate desserts, Brett has put rat poison into the cherries flambé; Margo, unaware of Brett's activities, has laced the chocolate mousse with cyanide. Given the amounts likely to be eaten, the probability of the rat poison being fatal is 0.60; the cyanide, 0.90. Based on other dinners where Basil was presented with the same dessert options, we can assume that he has a 50% chance of asking for the cherries flambé, a 40% chance of ordering the chocolate mousse, and a 10% chance of skipping dessert altogether. No sooner are the dishes cleared away than Basil drops dead. In the absence of any other evidence, who should be considered the prime suspect?

2.4.47. Josh takes a twenty-question multiple-choice exam where each question has five possible answers. Some of the answers he knows, while others he gets right just by making lucky guesses. Suppose that the conditional probability of his knowing the answer to a randomly selected question given that he got it right is 0.92. How many of the twenty questions was he prepared for?

2.4.48. Recently the U.S. Senate Committee on Labor and Public Welfare investigated the feasibility of setting up a national screening program to detect child abuse. A team of consultants estimated the following probabilities: (1) one child in ninety is abused, (2) a screening program can detect an abused child 90% of the time, and (3) a screening program would incorrectly label 3% of all nonabused children as abused. What is the probability that a child is actually abused given that the screening program makes that diagnosis? How does the probability change if the incidence of abuse is one in one thousand? Or one in fifty?

2.4.49. At State University, 30% of the students are majoring in humanities, 50% in history and culture, and 20% in science. Moreover, according to figures released by the registrar, the percentages of women majoring in humanities, history and culture, and science are 75%, 45%, and 30%, respectively. Suppose Justin meets Anna at a fraternity party. What is the probability that Anna is a history and culture major?

2.4.50. An "eyes-only" diplomatic message is to be transmitted as a binary code of 0's and 1's. Past experience with the equipment being used suggests that if a 0 is sent, it will be (correctly) received as a 0 90% of the time (and mistakenly decoded as a 1 10% of the time). If a 1 is sent, it will be received as a 1 95% of the time (and as a 0 5% of the time). The text being sent is thought to be 70% 1's and 30% 0's. Suppose the next signal sent is received as a 1. What is the probability that it was sent as a 0?

2.4.51. When Zach wants to contact his girlfriend and he knows she is not at home, he is twice as likely to send her an e-mail as he is to leave a message on her answering machine. The probability that she responds to his e-mail within three hours is 80%; her chances of being similarly prompt in answering a phone message increase to 90%. Suppose she responded within two hours to the message he left this morning. What is the probability that Zach was communicating with her via e-mail?

2.4.52. A dot-com company ships products from three different warehouses (A, B, and C). Based on customer complaints, it appears that 3% of the shipments coming from A are somehow faulty, as are 5% of the shipments coming from B, and 2% coming from C. Suppose a customer is mailed an order and calls in a complaint the next day. What is the probability the item came from Warehouse C? Assume that Warehouses A, B, and C ship 30%, 20%, and 50% of the dot-com's sales, respectively.

2.4.53. A desk has three drawers. The first contains two gold coins, the second has two silver coins, and the third has one gold coin and one silver coin. A coin is drawn from a drawer selected at random. Suppose the coin selected was silver. What is the probability that the other coin in that drawer is gold?

2.5 Independence

Section 2.4 dealt with the problem of reevaluating the probability of a given event in light of the additional information that some other event has already occurred. It often is the case, though, that the probability of the given event remains unchanged, regardless of the outcome of the second event—that is, $P(A|B) = P(A) = P(A|B^C)$. Events sharing this property are said to be *independent*. Definition 2.5.1 gives a necessary and sufficient condition for two events to be independent.

Definition 2.5.1. Two events A and B are said to be *independent* if $P(A \cap B) = P(A) \cdot P(B)$.

$$\begin{aligned}P(\text{Emma and Josh have different blood types}) &= 1 - \{P(E_A)P(J_A) + P(E_B)P(J_B) \\ &\quad + P(E_{AB})P(J_{AB}) + P(E_O)P(J_O)\} \\ &= 1 - \{(0.40)(0.40) + (0.10)(0.10) \\ &\quad + (0.05)(0.05) + (0.45)(0.45)\} \\ &= 0.625\end{aligned}$$

Questions

2.5.1. Suppose that $P(A \cap B) = 0.2$, $P(A) = 0.6$, and $P(B) = 0.5$.

(a) Are A and B mutually exclusive?
(b) Are A and B independent?
(c) Find $P(A^C \cup B^C)$.

2.5.2. Spike is not a terribly bright student. His chances of passing chemistry are 0.35; mathematics, 0.40; and both, 0.12. Are the events "Spike passes chemistry" and "Spike passes mathematics" independent? What is the probability that he fails both subjects?

2.5.3. Two fair dice are rolled. What is the probability that the number showing on one will be twice the number appearing on the other?

2.5.4. Urn I has three red chips, two black chips, and five white chips; urn II has two red, four black, and three white. One chip is drawn at random from each urn. What is the probability that both chips are the same color?

2.5.5. Dana and Cathy are playing tennis. The probability that Dana wins at least one out of two games is 0.3. What is the probability that Dana wins at least one out of four?

2.5.6. Three points, X_1, X_2, and X_3, are chosen at random in the interval $(0, a)$. A second set of three points, Y_1, Y_2, and Y_3, are chosen at random in the interval $(0, b)$. Let A be the event that X_2 is between X_1 and X_3. Let B be the event that $Y_1 < Y_2 < Y_3$. Find $P(A \cap B)$.

2.5.7. Suppose that $P(A) = \frac{1}{4}$ and $P(B) = \frac{1}{8}$.

(a) What does $P(A \cup B)$ equal if
  1. A and B are mutually exclusive?
  2. A and B are independent?
(b) What does $P(A \mid B)$ equal if
  1. A and B are mutually exclusive?
  2. A and B are independent?

2.5.8. Suppose that events A, B, and C are independent.

(a) Use a Venn diagram to find an expression for $P(A \cup B \cup C)$ that does *not* make use of a complement.
(b) Find an expression for $P(A \cup B \cup C)$ that *does* make use of a complement.

2.5.9. A fair coin is tossed four times. What is the probability that the number of heads appearing on the first two tosses is equal to the number of heads appearing on the second two tosses?

2.5.10. Suppose that two cards are drawn simultaneously from a standard 52-card poker deck. Let A be the event that both are either a jack, queen, king, or ace of hearts, and let B be the event that both are aces. Are A and B independent? (*Note*: There are 1326 equally likely ways to draw two cards from a poker deck.)

Defining the Independence of More Than Two Events

It is not immediately obvious how to extend Definition 2.5.1 to, say, *three* events. To call A, B, and C independent, should we require that the probability of the three-way intersection factors into the product of the three original probabilities,

$$P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C) \tag{2.5.3}$$

$$= (0.5)(0.3) + (0.5)(0.7)(0.5)(0.3) + (0.5)(0.7)(0.5)(0.7)(0.5)(0.3) + \cdots$$

$$= (0.5)(0.3)\sum_{k=0}^{\infty}(0.35)^k$$

$$= (0.15)\left(\frac{1}{1-0.35}\right) = \frac{3}{13}$$

Now consider the second scenario. If Andy shoots at Bob and misses, Bob will undoubtedly shoot and hit Charley, since Charley is the more dangerous adversary. Then it will be Andy's turn again. Whether he would see another tomorrow would depend on his ability to make that very next shot count. Specifically,

$$P(\text{Andy survives}) = P(\text{Andy hits Bob on second turn}) = 3/10$$

But $\frac{3}{10} > \frac{3}{13}$, so Andy is better off *not* hitting Bob with his first shot. And because we have already argued that it would be foolhardy for Andy to shoot at Charley, Andy's optimal strategy is clear—deliberately miss both Bob and Charley with the first shot. ■

Questions

2.5.11. Suppose that two fair dice (one red and one green) are rolled. Define the events

A: a 1 or a 2 shows on the red die
B: a 3, 4, or 5 shows on the green die
C: the dice total is 4, 11, or 12

Show that these events satisfy Equation 2.5.3 but not Equation 2.5.4.

2.5.12. A roulette wheel has thirty-six numbers colored red or black according to the pattern indicated below:

Roulette wheel pattern

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
R	R	R	R	R	B	B	B	B	R	R	R	R	B	B	B	B	B
36	35	34	33	32	31	30	29	28	27	26	25	24	23	22	21	20	19

Define the events

A: red number appears
B: even number appears
C: number is less than or equal to 18

Show that these events satisfy Equation 2.5.4 but not Equation 2.5.3.

2.5.13. How many probability equations need to be verified to establish the mutual independence of *four* events?

2.5.14. In a roll of a pair of fair dice (one red and one green), let A be the event the red die shows a 3, 4, or 5; let B be the event the green die shows a 1 or a 2; and let C be the event the dice total is 7. Show that A, B, and C are independent.

2.5.15. In a roll of a pair of fair dice (one red and one green), let A be the event of an odd number on the red die, let B be the event of an odd number on the green die, and let C be the event that the sum is odd. Show that any pair of these events is independent but that A, B, and C are not mutually independent.

2.5.16. On her way to work, a commuter encounters four traffic signals. Assume that the distance between each of the four is sufficiently great that her probability of getting a green light at any intersection is independent of what happened at any previous intersection. The first two lights are green for forty seconds of each minute; the last two, for thirty seconds of each minute. What is the probability that the commuter has to stop at least three times?

2.5.17. School board officials are debating whether to require all high school seniors to take a proficiency exam before graduating. A student passing all three parts (mathematics, language skills, and general knowledge) would be awarded a diploma; otherwise, he or she would receive only a certificate of attendance. A practice test given to this year's ninety-five hundred seniors resulted in the following numbers of failures:

Subject Area	Number of Students Failing
Mathematics	3325
Language skills	1900
General knowledge	1425

If "Student fails mathematics," "Student fails language skills," and "Student fails general knowledge" are independent events, what proportion of next year's seniors can