feasible. It often *is* feasible, though, to construct a simple, but analogous, problem for which the entire set of outcomes can be identified (and counted). If the proposed formula does not agree with the simple-case enumeration, we know that our analysis of the original question is incorrect.

3. If the outcomes to be counted fall into structurally different categories, the total number of outcomes will be the *sum* (not the product) of the number of outcomes in each category. Recall Example 2.6.5. The categories there are the nine different name lengths.

Questions

2.6.1. A chemical engineer wishes to observe the effects of temperature, pressure, and catalyst concentration on the yield resulting from a certain reaction. If she intends to include two different temperatures, three pressures, and two levels of catalyst, how many different runs must she make in order to observe each temperature-pressure-catalyst combination exactly twice?

2.6.2. A coded message from a CIA operative to his Russian KGB counterpart is to be sent in the form Q4ET, where the first and last entries must be consonants; the second, an integer 1 through 9; and the third, one of the six vowels. How many different ciphers can be transmitted?

2.6.3. How many terms will be included in the expansion of

$$(a+b+c)(d+e+f)(x+y+u+v+w)$$

Which of the following will be included in that number: *aeu, cdx, bef, xvw*?

2.6.4. Suppose that the format for license plates in a certain state is two letters followed by four numbers.

- **(a)** How many different plates can be made?
- **(b)** How many different plates are there if the letters can be repeated but no two numbers can be the same?
- **(c)** How many different plates can be made if repetitions of numbers and letters are allowed except that no plate can have four zeros?

2.6.5. How many integers between 100 and 999 have distinct digits, and how many of those are odd numbers?

2.6.6. A fast-food restaurant offers customers a choice of eight toppings that can be added to a hamburger. How many different hamburgers can be ordered?

2.6.7. In baseball there are twenty-four different "base-out" configurations (runner on first—two outs, bases loaded—none out, and so on). Suppose that a new game, sleazeball, is played where there are seven bases (excluding home plate) and each team gets five outs an inning. How many base-out configurations would be possible in sleazeball?

2.6.8. When they were first introduced, postal zip codes were five-digit numbers, theoretically ranging from 00000 to 99999. (In reality, the lowest zip code was 00601 for San Juan, Puerto Rico; the highest was 99950 for Ketchikan, Alaska.) An additional four digits have been added, so each zip code is now a nine-digit number. How many zip codes are at least as large as 60000–0000, are even numbers, and have a 7 as their third digit?

2.6.9. A restaurant offers a choice of four appetizers, fourteen entrees, six desserts, and five beverages. How many different meals are possible if a diner intends to order only three courses? (Consider the beverage to be a "course.")

2.6.10. An octave contains twelve distinct notes (on a piano, five black keys and seven white keys). How many different eight-note melodies within a single octave can be written if the black keys and white keys need to alternate?

2.6.11. Residents of a condominium have an automatic garage door opener that has a row of eight buttons. Each garage door has been programmed to respond to a particular set of buttons being pushed. If the condominium houses 250 families, can residents be assured that no two garage doors will open on the same signal? If so, how many additional families can be added before the eight-button code becomes inadequate? (*Note:* The order in which the buttons are pushed is irrelevant.)

2.6.12. In international Morse code, each letter in the alphabet is symbolized by a series of dots and dashes: the letter *a*, for example, is encoded as "· –". What is the minimum number of dots and/or dashes needed to represent any letter in the English alphabet?

2.6.13. The decimal number corresponding to a sequence of n binary digits $a_0, a_1, \ldots, a_{n-1}$, where each a_i is either 0 or 1, is defined to be

$$a_0 2^0 + a_1 2^1 + \cdots + a_{n-1} 2^{n-1}$$

For example, the sequence 0 1 1 0 is equal to 6 ($= 0 \cdot 2^0 + 1 \cdot 2^1 + 1 \cdot 2^2 + 0 \cdot 2^3$). Suppose a fair coin is tossed nine times. Replace the resulting sequence of H's and

and the 2 precede the 3 and the 4? That is, we want to count sequences like 7 *2* 5 *1 3* 6 9 *4* 8 but not like 6 8 *1* 5 *4 2* 7 *3* 9.

At first glance, this seems to be a problem well beyond the scope of Theorem 2.6.1. With the help of a symmetry argument, though, its solution is surprisingly simple.

Think of just the digits 1 through 4. By the corollary on p. 74, those four numbers give rise to $4!(=24)$ permutations. Of those twenty-four, only four—(1, 2, 3, 4), (2, 1, 3, 4), (1, 2, 4, 3), and (2, 1, 4, 3)—have the property that the 1 and the 2 come before the 3 and the 4. It follows that $\frac{4}{24}$ of the total number of nine-digit permutations should satisfy the condition being imposed on 1, 2, 3, and 4. Therefore,

$$\begin{aligned}\text{number of permutations where 1 and 2 precede 3 and 4} &= \frac{4}{24}\cdot 9!\\ &= 60{,}480\end{aligned}$$

■

Questions

2.6.17. The board of a large corporation has six members willing to be nominated for office. How many different "president/vice president/treasurer" slates could be submitted to the stockholders?

2.6.18. How many ways can a set of four tires be put on a car if all the tires are interchangeable? How many ways are possible if two of the four are snow tires?

2.6.19. Use Stirling's formula to approximate 30!. (*Note:* The exact answer is 265,252,859,812,268,935,315,188, 480,000,000.)

2.6.20. The nine members of the music faculty baseball team, the Mahler Maulers, are all incompetent, and each can play any position equally poorly. In how many different ways can the Maulers take the field?

2.6.21. A three-digit number is to be formed from the digits 1 through 7, with no digit being used more than once. How many such numbers would be less than 289?

2.6.22. Four men and four women are to be seated in a row of chairs numbered 1 through 8.

- **(a)** How many total arrangements are possible?
- **(b)** How many arrangements are possible if the men are required to sit in alternate chairs?

2.6.23. An engineer needs to take three technical electives sometime during his final four semesters. The three are to be selected from a list of ten. In how many ways can he schedule those classes, assuming that he never wants to take more than one technical elective in any given term?

2.6.24. How many ways can a twelve-member cheerleading squad (six men and six women) pair up to form six male-female teams? How many ways can six male-female teams be positioned along a sideline? What might the number $6!6!2^6$ represent? What might the number $6!6!2^62^{12}$ represent?

2.6.25. Suppose that a seemingly interminable German opera is recorded on all six sides of a three-record album. In how many ways can the six sides be played so that at least one is out of order?

2.6.26. A group of n families, each with m members, are to be lined up for a photograph. In how many ways can the nm people be arranged if members of a family must stay together?

2.6.27. Suppose that ten people, including you and a friend, line up for a group picture. How many ways can the photographer rearrange the line if she wants to keep exactly three people between you and your friend?

2.6.28. Use an induction argument to prove Theorem 2.6.1. (*Note*: This was the first mathematical result known to have been proved by induction. It was done in 1321 by Levi ben Gerson.)

2.6.29. In how many ways can a pack of fifty-two cards be dealt to thirteen players, four to each, so that every player has one card of each suit?

2.6.30. If the definition of $n!$ is to hold for all nonnegative integers n, show that it follows that 0! must equal 1.

2.6.31. The crew of Apollo 17 consisted of a pilot, a copilot, and a geologist. Suppose that NASA had actually trained nine aviators and four geologists as candidates for the flight. How many different crews could they have assembled?

Guessing at the rate of one permutation every five seconds would allow 360 permutations to be tested in thirty minutes, but 360 is only 63% of 570, so the burglar's 70% probability criteria of success would not be met. (*Question*: The first factors in Column 3 of Table 2.6.3 are applications of Theorem 2.6.2 to the sample permutations shown in Column 2. What do the second factors in Column 3 represent?) ■

Questions

2.6.34. Which state name can generate more permutations, TENNESSEE or FLORIDA?

2.6.35. How many numbers greater than four million can be formed from the digits 2, 3, 4, 4, 5, 5, 5?

2.6.36. An interior decorator is trying to arrange a shelf containing eight books, three with red covers, three with blue covers, and two with brown covers.

(a) Assuming the titles and the sizes of the books are irrelevant, in how many ways can she arrange the eight books?
(b) In how many ways could the books be arranged if they were all considered distinct?
(c) In how many ways could the books be arranged if the red books were considered indistinguishable, but the other five were considered distinct?

2.6.37. Four Nigerians (A, B, C, D), three Chinese (#, *, &), and three Greeks (α, β, γ) are lined up at the box office, waiting to buy tickets for the World's Fair.

(a) How many ways can they position themselves if the Nigerians are to hold the first four places in line; the Chinese, the next three; and the Greeks, the last three?
(b) How many arrangements are possible if members of the same nationality must stay together?
(c) How many different queues can be formed?
(d) Suppose a vacationing Martian strolls by and wants to photograph the ten for her scrapbook. A bit myopic, the Martian is quite capable of discerning the more obvious differences in human anatomy but is unable to distinguish one Nigerian (N) from another, one Chinese (C) from another, or one Greek (G) from another. Instead of perceiving a line to be $B*\beta AD\#\&C\alpha\gamma$, for example, she would see *NCGNNCCNGG*. From the Martian's perspective, in how many different ways can the ten funny-looking Earthlings line themselves up?

2.6.38. How many ways can the letters in the word

S L U M G U L L I O N

be arranged so that the three L's precede all the other consonants?

2.6.39. A tennis tournament has a field of $2n$ entrants, all of whom need to be scheduled to play in the first round. How many different pairings are possible?

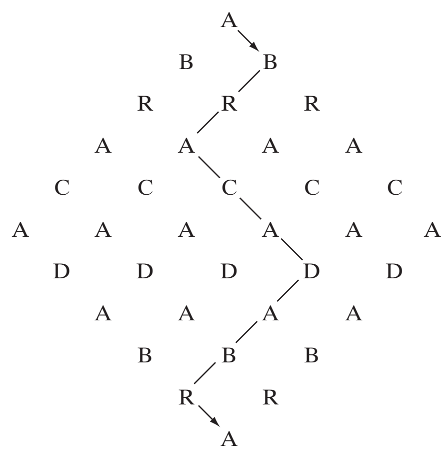
2.6.40. What is the coefficient of x^{12} in the expansion of $(1+x^3+x^6)^{18}$?

2.6.41. In how many ways can the letters of the word

E L E E M O S Y N A R Y

be arranged so that the S is always immediately followed by a Y?

2.6.42. In how many ways can the word *ABRACADABRA* be formed in the array pictured below? Assume that the word must begin with the top A and progress diagonally downward to the bottom A.



2.6.43. Suppose a pitcher faces a batter who never swings. For how many different ball/strike sequences will the batter be called out on the fifth pitch?

2.6.44. What is the coefficient of $w^2x^3yz^3$ in the expansion of $(w+x+y+z)^9$?

2.6.45. Imagine six points in a plane, no three of which lie on a straight line. In how many ways can the six points be used as vertices to form two triangles? (*Hint:* Number the points 1 through 6. Call one of the triangles A and the other B. What does the permutation

Problem-Solving Hints

(Doing combinatorial probability problems)

Listed on p. 72 are several hints that can be helpful in counting the number of ways to do something. Those same hints apply to the solution of combinatorial *probability* problems, but a few others should be kept in mind as well.

1. The solution to a combinatorial probability problem should be set up as a quotient of numerator and denominator *enumerations*. Avoid the temptation to multiply probabilities associated with each position in the sequence. The latter approach will always "sound" reasonable, but it will frequently oversimplify the problem and give the wrong answer.
2. Keep the numerator and denominator consistent with respect to *order*—if permutations are being counted in the numerator, be sure that permutations are being counted in the denominator; likewise, if the outcomes in the numerator are combinations, the outcomes in the denominator must also be combinations.
3. The number of outcomes associated with any problem involving the rolling of n six-sided dice is 6^n; similarly, the number of outcomes associated with tossing a coin n times is 2^n. The number of outcomes associated with dealing a hand of n cards from a standard 52-card poker deck is ${}_{52}C_n$.

Questions

2.7.1. Ten equally qualified marketing assistants are candidates for promotion to associate buyer; seven are men and three are women. If the company intends to promote four of the ten at random, what is the probability that exactly two of the four are women?

2.7.2. An urn contains six chips, numbered 1 through 6. Two are chosen at random and their numbers are added together. What is the probability that the resulting sum is equal to 5?

2.7.3. An urn contains twenty chips, numbered 1 through 20. Two are drawn simultaneously. What is the probability that the numbers on the two chips will differ by more than 2?

2.7.4. A bridge hand (thirteen cards) is dealt from a standard 52-card deck. Let A be the event that the hand contains four aces; let B be the event that the hand contains four kings. Find $P(A \cup B)$.

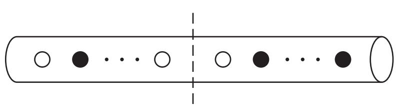
2.7.5. Consider a set of ten urns, nine of which contain three white chips and three red chips each. The tenth contains five white chips and one red chip. An urn is picked at random. Then a sample of size 3 is drawn without replacement from that urn. If all three chips drawn are white, what is the probability that the urn being sampled is the one with five white chips?

2.7.6. A committee of fifty politicians is to be chosen from among our one hundred U.S. senators. If the selection is done at random, what is the probability that each state will be represented?

2.7.7. Suppose that n fair dice are rolled. What are the chances that all n faces will be the same?

2.7.8. Five fair dice are rolled. What is the probability that the faces showing constitute a "full house"—that is, three faces show one number and two faces show a second number?

2.7.9. Imagine that the test tube pictured contains $2n$ grains of sand, n white and n black. Suppose the tube is vigorously shaken. What is the probability that the two colors of sand will completely separate; that is, all of one color fall to the bottom, and all of the other color lie on top? (*Hint:* Consider the $2n$ grains to be aligned in a row. In how many ways can the n white and the n black grains be permuted?)



2.7.10. Does a monkey have a better chance of rearranging

$$A\,C\,C\,L\,L\,U\,U\,S \quad \text{to spell} \quad C\,A\,L\,C\,U\,L\,U\,S$$

or

$$A\,A\,B\,E\,G\,L\,R \quad \text{to spell} \quad A\,L\,G\,E\,B\,R\,A?$$

2.7.11. An apartment building has eight floors. If seven people get on the elevator on the first floor, what is the probability they all want to get off on different floors? On the same floor? What assumption are you making? Does it seem reasonable? Explain.

2.7.12. If the letters in the phrase

$$A\ \ R\,O\,L\,L\,I\,N\,G\ \ S\,T\,O\,N\,E\ \ G\,A\,T\,H\,E\,R\,S\ \ N\,O\ \ M\,O\,S\,S$$

are arranged at random, what are the chances that not all the S's will be adjacent?

2.7.13. Suppose each of ten sticks is broken into a long part and a short part. The twenty parts are arranged into ten pairs and glued back together so that again there are ten sticks. What is the probability that each long part will be paired with a short part? (*Note:* This problem is a model for the effects of radiation on a living cell. Each chromosome, as a result of being struck by ionizing radiation, breaks into two parts, one part containing the centromere. The cell will die unless the fragment containing the centromere recombines with a fragment not containing a centromere.)

2.7.14. Six dice are rolled one time. What is the probability that each of the six faces appears?

2.7.15. Suppose that a randomly selected group of k people are brought together. What is the probability that exactly one pair has the same birthday?

2.7.16. For one-pair poker hands, why is the number of denominations for the three single cards $\binom{12}{3}$ rather than $\binom{12}{1}\binom{11}{1}\binom{10}{1}$?

2.7.17. Dana is not the world's best poker player. Dealt a 2 of diamonds, an 8 of diamonds, an ace of hearts, an ace of clubs, and an ace of spades, she discards the three aces. What are her chances of drawing to a flush?

2.7.18. A poker player is dealt a 7 of diamonds, a queen of diamonds, a queen of hearts, a queen of clubs, and an ace of hearts. He discards the 7. What is his probability of drawing to either a full house or four-of-a-kind?

2.7.19. Tim is dealt a 4 of clubs, a 6 of hearts, an 8 of hearts, a 9 of hearts, and a king of diamonds. He discards the 4 and the king. What are his chances of drawing to a straight flush? To a flush?

2.7.20. Five cards are dealt from a standard 52-card deck. What is the probability that the sum of the faces on the five cards is 48 or more?

2.7.21. Nine cards are dealt from a 52-card deck. Write a formula for the probability that three of the five even numerical denominations are represented twice, one of the three face cards appears twice, and a second face card appears once. (*Note:* Face cards are the jacks, queens, and kings; 2, 4, 6, 8, and 10 are the even numerical denominations.)

2.7.22. A coke hand in bridge is one where none of the thirteen cards is an ace or is higher than a 9. What is the probability of being dealt such a hand?

2.7.23. A pinochle deck has forty-eight cards, two of each of six denominations (9, J, Q, K, 10, A) and the usual four suits. Among the many hands that count for meld is a *roundhouse*, which occurs when a player has a king and queen of each suit. In a hand of twelve cards, what is the probability of getting a "bare" roundhouse (a king and queen of each suit and no other kings or queens)?

2.7.24. A somewhat inebriated conventioneer finds himself in the embarrassing predicament of being unable to predetermine whether his next step will be forward or backward. What is the probability that after hazarding n such maneuvers he will have stumbled forward a distance of r steps? (*Hint:* Let x denote the number of steps he takes forward and y, the number backward. Then $x + y = n$ and $x - y = r$.)

2.8 Taking a Second Look at Statistics (Monte Carlo Techniques)

Recall the von Mises definition of probability given on p. 17. If an experiment is repeated n times under identical conditions, and if the event E occurs on m of those repetitions, then

$$P(E) = \lim_{n\to\infty} \frac{m}{n} \tag{2.8.1}$$