More specifically, the median lifetime for these bulbs—according to Definition 3.5.2—is the value m for which

$$\int_0^m 0.001e^{-0.001y}\,dy = 0.5$$

But $\int_0^m 0.001e^{-0.001y}\,dy = 1 - e^{-0.001m}$. Setting the latter equal to 0.5 implies that

$$m = (1/-0.001)\ln(0.5) = 693$$

So, even though the *average* life of one of these bulbs is 1000 hours, there is a 50% chance that the one you buy will last less than 693 hours. ■

Questions

3.5.1. Recall the game of Keno described in Question 3.2.26. The following are all the payoffs on a \$1 wager where the player has bet on ten numbers. Calculate $E(X)$, where the random variable X denotes the amount of money won.

Number of Correct Guesses	Payoff	Probability
< 5	−\$ 1	.935
5	2	.0514
6	18	.0115
7	180	.0016
8	1,300	1.35×10^{-4}
9	2,600	6.12×10^{-6}
10	10,000	1.12×10^{-7}

3.5.2. The roulette wheels in Monte Carlo typically have a 0 but not a 00. What is the expected value of betting on red in this case? If a trip to Monte Carlo costs \$3000, how much would a player have to bet to justify gambling there rather than Las Vegas?

3.5.3. The pdf describing the daily profit, X, earned by Acme Industries was derived in Example 3.3.7. Find the company's *average* daily profit.

3.5.4. In the game of redball, two drawings are made without replacement from a bowl that has four white ping-pong balls and two red ping-pong balls. The amount won is determined by how many of the red balls are selected. For a \$5 bet, a player can opt to be paid under either Rule A or Rule B, as shown. If you were playing the game, which would you choose? Why?

A		*B*	
No. of Red Balls Drawn	Payoff	No. of Red Balls Drawn	Payoff
0	0	0	0
1	\$2	1	\$1
2	\$10	2	\$20

3.5.5. Suppose a life insurance company sells a \$50,000, five-year term policy to a twenty-five-year-old woman. At the beginning of each year the woman is alive, the company collects a premium of \$$P$. The probability that the woman dies and the company pays the \$50,000 is given in the table below. So, for example, in Year 3, the company loses \$50,000 – \$$P$ with probability 0.00054 and gains \$$P$ with probability 1 – 0.00054 = 0.99946. If the company expects to make \$1000 on this policy, what should P be?

Year	Probability of Payoff
1	0.00051
2	0.00052
3	0.00054
4	0.00056
5	0.00059

3.5.6. A manufacturer has one hundred memory chips in stock, 4% of which are likely to be defective (based on past experience). A random sample of twenty chips is selected and shipped to a factory that assembles laptops. Let X denote the number of computers that receive faulty memory chips. Find $E(X)$.

3.5.7. Records show that 642 new students have just entered a certain Florida school district. Of those 642, a total of 125 are not adequately vaccinated. The district's physician has scheduled a day for students to receive whatever shots they might need. On any given day, though, 12% of the district's students are likely to be absent. How many new students, then, can be expected to remain inadequately vaccinated?

3.5.8. Calculate $E(Y)$ for the following pdfs:

(a) $f_Y(y) = 3(1-y)^2,\ 0 \le y \le 1$

(b) $f_Y(y) = 4ye^{-2y},\ y \ge 0$

(c) $f_Y(y) = \begin{cases} \frac{3}{4}, & 0 \le y \le 1 \\ \frac{1}{4}, & 2 \le y \le 3 \\ 0, & \text{elsewhere} \end{cases}$

(d) $f_Y(y) = \sin y,\quad 0 \le y \le \frac{\pi}{2}$

3.5.9. Recall Question 3.4.4, where the length of time Y (in years) that a malaria patient spends in remission has pdf $f_Y(y) = \frac{1}{9}y^2, 0 \leq y \leq 3$. What is the average length of time that such a patient spends in remission?

3.5.10. Let the random variable Y have the uniform distribution over $[a, b]$; that is, $f_Y(y) = \frac{1}{b-a}$ for $a \leq y \leq b$. Find $E(Y)$ using Definition 3.5.1. Also, deduce the value of $E(Y)$, knowing that the expected value is the center of gravity of $f_Y(y)$.

3.5.11. Show that the expected value associated with the exponential distribution, $f_Y(y) = \lambda e^{-\lambda y}$, $y > 0$, is $1/\lambda$, where λ is a positive constant.

3.5.12. Show that

$$f_Y(y) = \frac{1}{y^2}, \quad y \geq 1$$

is a valid pdf but that Y does not have a finite expected value.

3.5.13. Based on recent experience, ten-year-old passenger cars going through a motor vehicle inspection station have an 80% chance of passing the emissions test. Suppose that two hundred such cars will be checked out next week. Write two formulas that show the number of cars that are expected to pass.

3.5.14. Suppose that fifteen observations are chosen at random from the pdf $f_Y(y) = 3y^2, 0 \leq y \leq 1$. Let X denote the number that lie in the interval $\left(\frac{1}{2}, 1\right)$. Find $E(X)$.

3.5.15. A city has 74,806 registered automobiles. Each is required to display a bumper decal showing that the owner paid an annual wheel tax of \$50. By law, new decals need to be purchased during the month of the owner's birthday. How much wheel tax revenue can the city expect to receive in November?

3.5.16. Regulators have found that twenty-three of the sixty-eight investment companies that filed for bankruptcy in the past five years failed because of fraud, not for reasons related to the economy. Suppose that nine additional firms will be added to the bankruptcy rolls during the next quarter. How many of those failures are likely to be attributed to fraud?

3.5.17. An urn contains four chips numbered 1 through 4. Two are drawn without replacement. Let the random variable X denote the larger of the two. Find $E(X)$.

3.5.18. A fair coin is tossed three times. Let the random variable X denote the total number of heads that appear times the number of heads that appear on the first and third tosses. Find $E(X)$.

3.5.19. How much would you have to ante to make the St. Petersburg game "fair" (recall Example 3.5.5) if the most you could win was \$1000? That is, the payoffs are $\$2^k$ for $1 \leq k \leq 9$, and \$1000 for $k \geq 10$.

3.5.20. For the St. Petersburg problem (Example 3.5.5), find the expected payoff if

(a) the amounts won are c^k instead of 2^k, where $0 < c < 2$.
(b) the amounts won are $\log 2^k$. [This was a modification suggested by D. Bernoulli (a nephew of James Bernoulli) to take into account the decreasing marginal utility of money—the more you have, the less useful a bit more is.]

3.5.21. A fair die is rolled three times. Let X denote the number of different faces showing, $X = 1, 2, 3$. Find $E(X)$.

3.5.22. Two distinct integers are chosen at random from the first five positive integers. Compute the expected value of the absolute value of the difference of the two numbers.

3.5.23. Suppose that two evenly matched teams are playing in the World Series. On the average, how many games will be played? (The winner is the first team to get four victories.) Assume that each game is an independent event.

3.5.24. An urn contains one white chip and one black chip. A chip is drawn at random. If it is white, the "game" is over; if it is black, that chip and another black one are put into the urn. Then another chip is drawn at random from the "new" urn and the same rules for ending or continuing the game are followed (i.e., if the chip is white, the game is over; if the chip is black, it is placed back in the urn, together with another chip of the same color). The drawings continue until a white chip is selected. Show that the expected number of drawings necessary to get a white chip is not finite.

3.5.25. A random sample of size n is drawn without replacement from an urn containing r red chips and w white chips. Define the random variable X to be the number of red chips in the sample. Use the summation technique described in Theorem 3.5.1 to prove that $E(X) = rn/(r+w)$.

3.5.26. Given that X is a nonnegative, integer-valued random variable, show that

$$E(X) = \sum_{k=1}^{\infty} P(X \geq k)$$

3.5.27. Find the median for each of the following pdfs:

(a) $f_Y(y) = (\theta + 1)y^\theta, \ 0 \leq y \leq 1$, where $\theta > 0$
(b) $f_Y(y) = y + \frac{1}{2}, \ 0 \leq y \leq 1$

Questions

3.5.28. Suppose X is a binomial random variable with $n = 10$ and $p = \frac{2}{5}$. What is the expected value of $3X - 4$?

3.5.29. A typical day's production of a certain electronic component is twelve. The probability that one of these components needs rework is 0.11. Each component needing rework costs \$100. What is the average daily cost for defective components?

3.5.30. Let Y have probability density function

$$f_Y(y) = 2(1 - y),\ 0 \le y \le 1$$

Suppose that $W = Y^2$, in which case

$$f_W(w) = \frac{1}{\sqrt{w}} - 1,\ 0 \le w \le 1$$

Find $E(W)$ in two different ways.

3.5.31. A tool and die company makes castings for steel stress-monitoring gauges. Their annual profit, Q, in hundreds of thousands of dollars, can be expressed as a function of product demand, y:

$$Q(y) = 2(1 - e^{-2y})$$

Suppose that the demand (in thousands) for their castings follows an exponential pdf, $f_Y(y) = 6e^{-6y}$, $y > 0$. Find the company's expected profit.

3.5.32. A box is to be constructed so that its height is five inches and its base is Y inches by Y inches, where Y is a random variable described by the pdf, $f_Y(y) = 6y(1 - y), 0 < y < 1$. Find the expected volume of the box.

3.5.33. Grades on the last Economics 301 exam were not very good. Graphed, their distribution had a shape similar to the pdf

$$f_Y(y) = \frac{1}{5000}(100 - y), \quad 0 \le y \le 100$$

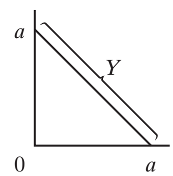
As a way of "curving" the results, the professor announces that he will replace each person's grade, Y, with a new grade, $g(Y)$, where $g(Y) = 10\sqrt{Y}$. Will the professor's strategy be successful in raising the class average above 60?

3.5.34. If Y has probability density function

$$f_Y(y) = 2y,\ 0 \le y \le 1$$

then $E(Y) = \frac{2}{3}$. Define the random variable W to be the squared deviation of Y from its mean, that is, $W = \left(Y - \frac{2}{3}\right)^2$. Find $E(W)$.

3.5.35. The hypotenuse, Y, of the isosceles right triangle shown is a random variable having a uniform pdf over the interval [6, 10]. Calculate the expected value of the triangle's area. Do not leave the answer as a function of a.



3.5.36. An urn contains n chips numbered 1 through n. Assume that the probability of choosing chip i is equal to $ki, i = 1, 2, \ldots, n$. If one chip is drawn, calculate $E\left(\frac{1}{X}\right)$, where the random variable X denotes the number showing on the chip selected. [*Hint*: Recall that the sum of the first n integers is $n(n + 1)/2$.]

3.6 The Variance

We saw in Section 3.5 that the location of a distribution is an important characteristic and that it can be effectively measured by calculating either the mean or the median. A second feature of a distribution that warrants further scrutiny is its *dispersion*—that is, the extent to which its values are spread out. The two properties are totally different: Knowing a pdf's location tells us absolutely nothing about its dispersion. Table 3.6.1, for example, shows two simple discrete pdfs with the same expected value (equal to zero), but with vastly different dispersions.

Table 3.6.1

k	$p_{X_1}(k)$	k	$p_{X_2}(k)$
-1	$\frac{1}{2}$	$-1{,}000{,}000$	$\frac{1}{2}$
1	$\frac{1}{2}$	1,000,000	$\frac{1}{2}$

so

$$\text{Var}(Y) = E(Y^2) - \mu^2 = \frac{1}{2} - \left(\frac{2}{3}\right)^2$$
$$= \frac{1}{18}$$

Then, by Theorem 3.6.2,

$$\text{Var}(3Y + 2) = (3)^2 \cdot \text{Var}(Y) = 9 \cdot \frac{1}{18}$$
$$= \frac{1}{2}$$

which makes the standard deviation of $3Y + 2$ equal to $\sqrt{\frac{1}{2}}$ or *0.71*. ■

Questions

3.6.1. Find $\text{Var}(X)$ for the urn problem of Example 3.6.1 if the sampling is done *with* replacement.

3.6.2. Find the variance of Y if

$$f_Y(y) = \begin{cases} \frac{3}{4}, & 0 \le y \le 1 \\ \frac{1}{4}, & 2 \le y \le 3 \\ 0, & \text{elsewhere} \end{cases}$$

3.6.3. Ten equally qualified applicants, six men and four women, apply for three lab technician positions. Unable to justify choosing any of the applicants over all the others, the personnel director decides to select the three at random. Let X denote the number of men hired. Compute the standard deviation of X.

3.6.4. Compute the variance for a uniform random variable defined on the unit interval.

3.6.5. Use Theorem 3.6.1 to find the variance of the random variable Y, where

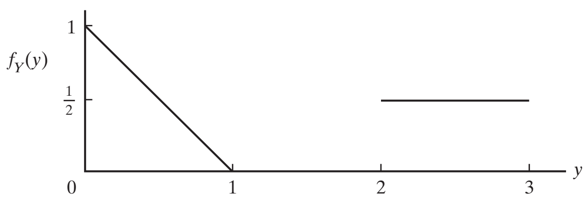
$$f_Y(y) = 3(1 - y)^2, \quad 0 < y < 1$$

3.6.6. If

$$f_Y(y) = \frac{2y}{k^2}, \quad 0 \le y \le k$$

for what value of k does $\text{Var}(Y) = 2$?

3.6.7. Calculate the standard deviation, σ, for the random variable Y whose pdf has the graph shown below:



3.6.8. Consider the pdf defined by

$$f_Y(y) = \frac{2}{y^3}, \quad y \ge 1$$

Show that (a) $\int_1^\infty f_Y(y)\,dy = 1$, (b) $E(Y) = 2$, and (c) $\text{Var}(Y)$ is not finite.

3.6.9. Frankie and Johnny play the following game. Frankie selects a number at random from the interval $[a, b]$. Johnny, not knowing Frankie's number, is to pick a second number from that same inverval and pay Frankie an amount, W, equal to the squared difference between the two [so $0 \le W \le (b - a)^2$]. What should be Johnny's strategy if he wants to minimize his expected loss?

3.6.10. Let Y be a random variable whose pdf is given by $f_Y(y) = 5y^4, 0 \le y \le 1$. Use Theorem 3.6.1 to find $\text{Var}(Y)$.

3.6.11. Suppose that Y is an exponential random variable, so $f_Y(y) = \lambda e^{-\lambda y}$, $y \ge 0$. Show that the variance of Y is $1/\lambda^2$.

3.6.12. Suppose that Y is an exponential random variable with $\lambda = 2$ (recall Question 3.6.11). Find $P[Y > E(Y) + 2\sqrt{\text{Var}(Y)}]$.

3.6.13. Let X be a random variable with finite mean μ. Define for every real number a, $g(a) = E[(X - a)^2]$. Show that

$$g(a) = E[(X - \mu)^2] + (\mu - a)^2.$$

What is another name for $\min g(a)$?

3.6.14. Let Y have the pdf given in Question 3.6.5. Find the variance of W, where $W = -5Y + 12$.

3.6.15. If Y denotes a temperature recorded in degrees Fahrenheit, then $\frac{5}{9}(Y - 32)$ is the corresponding temperature in degrees Celsius. If the standard deviation for a set of temperatures is 15.7°F, what is the standard deviation of the equivalent Celsius temperatures?

3.6.16. If $E(W) = \mu$ and $\text{Var}(W) = \sigma^2$, show that

$$E\left(\frac{W-\mu}{\sigma}\right) = 0 \quad \text{and} \quad \text{Var}\left(\frac{W-\mu}{\sigma}\right) = 1$$

3.6.17. Suppose U is a uniform random variable over [0, 1].

(a) Show that $Y = (b-a)U + a$ is uniform over $[a, b]$.
(b) Use part (a) and Question 3.6.4 to find the variance of Y.

3.6.18. Recovering small quantities of calcium in the presence of magnesium can be a difficult problem for an analytical chemist. Suppose the amount of calcium Y to be recovered is uniformly distributed between 4 and 7 mg. The amount of calcium recovered by one method is the random variable

$$W_1 = 0.2281 + (0.9948)Y + E_1$$

where the error term E_1 has mean 0 and variance 0.0427 and is independent of Y.

A second procedure has random variable

$$W_2 = -0.0748 + (1.0024)Y + E_2$$

where the error term E_2 has mean 0 and variance 0.0159 and is independent of Y.

The better technique should have a mean as close as possible to the mean of $Y (= 5.5)$, and a variance as small as possible. Compare the two methods on the basis of mean and variance.

Higher Moments

The quantities we have identified as the mean and the variance are actually special cases of what are referred to more generally as the *moments* of a random variable. More precisely, $E(W)$ is the *first moment about the origin* and σ^2 is the *second moment about the mean*. As the terminology suggests, we will have occasion to define higher moments of W. Just as $E(W)$ and σ^2 reflect a random variable's location and dispersion, so it is possible to characterize other aspects of a distribution in terms of other moments. We will see, for example, that the skewness of a distribution—that is, the extent to which it is not symmetric around μ—can be effectively measured in terms of a *third* moment. Likewise, there are issues that arise in certain applied statistics problems that require a knowledge of the flatness of a pdf, a property that can be quantified by the *fourth* moment.

Definition 3.6.2. Let W be any random variable with pdf $f_W(w)$. For any positive integer r,

1. The rth *moment of* W *about the origin*, μ_r, is given by

$$\mu_r = E(W^r)$$

provided $\int_{-\infty}^{\infty} |w|^r \cdot f_W(w)\, dw < \infty$ (or provided the analogous condition on the *summation* of $|w|^r$ holds, if W is discrete). When $r = 1$, we usually delete the subscript and write $E(W)$ as μ rather than μ_1.

2. The rth *moment of* W *about the mean*, μ_r', is given by

$$\mu_r' = E[(W-\mu)^r]$$

provided the finiteness conditions of part 1 hold.

Comment We can express μ_r' in terms of μ_j, $j = 1, 2, \ldots, r$, by simply writing out the binomial expansion of $(W-\mu)^r$:

$$\mu_r' = E[(W-\mu)^r] = \sum_{j=0}^{r} \binom{r}{j} E(W^j)(-\mu)^{r-j}$$

First, assume that the player is required to stand far enough away that no skill is involved and the ring is falling at random on the grid. From Figure 3.7.4, we see that in order for the ring not to touch any side of the square, the ring's center must be somewhere in the interior of a smaller square, each side of which is a distance $d/2$ from one of the grid lines.

Since the area of a grid square is s^2 and the area of an interior square is $(s-d)^2$, the probability of a winning toss can be written as the ratio:

$$P(\text{Ring touches no lines}) = \frac{(s-d)^2}{s^2}$$

But the operator requires that

$$\frac{(s-d)^2}{s^2} \le 0.20$$

Solving for d/s gives

$$\frac{d}{s} \ge 1 - \sqrt{0.20} = 0.55$$

That is, if the diameter of the ring is at least 55% as long as the side of one of the squares, the player will have no more than a 20% chance of winning. ■

Questions

3.7.1. If $p_{X,Y}(x, y) = cxy$ at the points (1, 1), (2, 1), (2, 2), and (3, 1), and equals 0 elsewhere, find c.

3.7.2. Let X and Y be two continuous random variables defined over the unit square. What does c equal if $f_{X,Y}(x, y) = c(x^2 + y^2)$?

3.7.3. Suppose that random variables X and Y vary in accordance with the joint pdf, $f_{X,Y}(x, y) = c(x + y)$, $0 < x < y < 1$. Find c.

3.7.4. Find c if $f_{X,Y}(x, y) = cxy$ for X and Y defined over the triangle whose vertices are the points (0, 0), (0, 1), and (1, 1).

3.7.5. An urn contains four red chips, three white chips, and two blue chips. A random sample of size 3 is drawn without replacement. Let X denote the number of white chips in the sample and Y the number of blue chips. Write a formula for the joint pdf of X and Y.

3.7.6. Four cards are drawn from a standard poker deck. Let X be the number of kings drawn and Y the number of queens. Find $p_{X,Y}(x, y)$.

3.7.7. An advisor looks over the schedules of his fifty students to see how many math and science courses each has registered for in the coming semester. He summarizes his results in a table. What is the probability that a student selected at random will have signed up for more math courses than science courses?

		Number of math courses, X		
		0	1	2
Number of science courses, Y	0	11	6	4
	1	9	10	3
	2	5	0	2

3.7.8. Consider the experiment of tossing a fair coin three times. Let X denote the number of heads on the last flip, and let Y denote the total number of heads on the three flips. Find $p_{X,Y}(x, y)$.

3.7.9. Suppose that two fair dice are tossed one time. Let X denote the number of 2's that appear, and Y the number of 3's. Write the matrix giving the joint probability density function for X and Y. Suppose a third random variable, Z, is defined, where $Z = X + Y$. Use $p_{X,Y}(x, y)$ to find $p_Z(z)$.

3.7.10. Suppose that X and Y have a bivariate uniform density over the unit square:

$$f_{X,Y}(x, y) = \begin{cases} c, & 0 < x < 1, \quad 0 < y < 1 \\ 0, & \text{elsewhere} \end{cases}$$

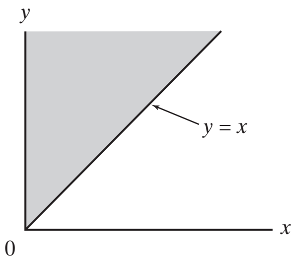
(a) Find c.
(b) Find $P\left(0 < X < \frac{1}{2}, 0 < Y < \frac{1}{4}\right)$.

3.7.20. For each of the following joint pdfs, find $f_X(x)$ and $f_Y(y)$.

(a) $f_{X,Y}(x, y) = \frac{1}{2}, 0 \le x \le y \le 2$
(b) $f_{X,Y}(x, y) = \frac{1}{x}, 0 \le y \le x \le 1$
(c) $f_{X,Y}(x, y) = 6x, 0 \le x \le 1, 0 \le y \le 1 - x$

3.7.21. Suppose that $f_{X,Y}(x, y) = 6(1 - x - y)$ for x and y defined over the unit square, subject to the restriction that $0 \le x + y \le 1$. Find the marginal pdf for X.

3.7.22. Find $f_Y(y)$ if $f_{X,Y}(x, y) = 2e^{-x}e^{-y}$ for x and y defined over the shaded region pictured.



3.7.23. Suppose that X and Y are discrete random variables with

$$p_{X,Y}(x, y) = \frac{4!}{x!y!(4-x-y)!}\left(\frac{1}{2}\right)^x\left(\frac{1}{3}\right)^y\left(\frac{1}{6}\right)^{4-x-y},$$
$$0 \le x + y \le 4$$

Find $p_X(x)$ and $p_Y(x)$.

3.7.24. A generalization of the binomial model occurs when there is a sequence of n independent trials with *three* outcomes, where $p_1 = P(\text{outcome } 1)$ and $p_2 = P(\text{outcome } 2)$. Let X and Y denote the number of trials (out of n) resulting in outcome 1 and outcome 2, respectively.

(a) Show that $p_{X,Y}(x, y) = \frac{n!}{x!y!(n-x-y)!}p_1^x p_2^y (1 - p_1 - p_2)^{n-x-y}, 0 \le x + y \le n$
(b) Find $p_X(x)$ and $p_Y(x)$.

(*Hint:* See Question 3.7.23.)

Joint Cdfs

For a single random variable X, the cdf of X evaluated at some point x—that is, $F_X(x)$—is the probability that the random variable X takes on a value less than or equal to x. Extended to two variables, a *joint cdf* [evaluated at the point (u, v)] is the probability that $X \le u$ and, simultaneously, that $Y \le v$.

Definition 3.7.4. Let X and Y be any two random variables. The *joint cumulative distribution function of X and Y* (or *joint cdf*) is denoted $F_{X,Y}(u, v)$, where

$$F_{X,Y}(u, v) = P(X \le u \quad \text{and} \quad Y \le v)$$

Example 3.7.9 Find the joint cdf, $F_{X,Y}(u, v)$, for the two random variables X and Y whose joint pdf is given by $f_{X,Y}(x, y) = \frac{4}{3}(x + xy), 0 \le x \le 1, 0 \le y \le 1$.

If Definition 3.7.4 is applied, the probability that $X \le u$ and $Y \le v$ becomes a double integral of $f_{X,Y}(x, y)$:

$$\begin{aligned}
F_{X,Y}(u, v) &= \frac{4}{3}\int_0^v \int_0^u (x + xy)\,dx\,dy = \frac{4}{3}\int_0^v \left[\int_0^u (x + xy)\,dx\right] dy \\
&= \frac{4}{3}\int_0^v \left[\frac{x^2}{2}(1 + y)\Big|_0^u\right] dy = \frac{4}{3}\int_0^v \frac{u^2}{2}(1 + y)\,dy \\
&= \frac{4}{3}\frac{u^2}{2}\left(y + \frac{y^2}{2}\right)\Big|_0^v = \frac{4}{3}\frac{u^2}{2}\left(v + \frac{v^2}{2}\right)
\end{aligned}$$

Questions

3.7.25. Consider the experiment of simultaneously tossing a fair coin and rolling a fair die. Let X denote the number of heads showing on the coin and Y the number of spots showing on the die.

(a) List the outcomes in S.
(b) Find $F_{X,Y}(1, 2)$.

3.7.26. An urn contains twelve chips—four red, three black, and five white. A sample of size 4 is to be drawn without replacement. Let X denote the number of white chips in the sample, Y the number of red. Find $F_{X,Y}(1, 2)$.

3.7.27. For each of the following joint pdfs, find $F_{X,Y}(u, v)$.

(a) $f_{X,Y}(x, y) = \frac{3}{2}y^2, 0 \le x \le 2, 0 \le y \le 1$
(b) $f_{X,Y}(x, y) = \frac{2}{3}(x + 2y), 0 \le x \le 1, 0 \le y \le 1$
(c) $f_{X,Y}(x, y) = 4xy, 0 \le x \le 1, 0 \le y \le 1$

3.7.28. For each of the following joint pdfs, find $F_{X,Y}(u, v)$.

(a) $f_{X,Y}(x, y) = \frac{1}{2}, 0 \le x \le y \le 2$
(b) $f_{X,Y}(x, y) = \frac{1}{x}, 0 \le y \le x \le 1$
(c) $f_{X,Y}(x, y) = 6x, 0 \le x \le 1, 0 \le y \le 1 - x$

3.7.29. Find and graph $f_{X,Y}(x, y)$ if the joint cdf for random variables X and Y is

$$F_{X,Y}(x, y) = xy, \qquad 0 < x < 1, \quad 0 < y < 1$$

3.7.30. Find the joint pdf associated with two random variables X and Y whose joint cdf is

$$F_{X,Y}(x, y) = (1 - e^{-\lambda y})(1 - e^{-\lambda x}), \qquad x > 0, \quad y > 0$$

3.7.31. Given that $F_{X,Y}(x, y) = k(4x^2y^2 + 5xy^4), 0 < x < 1, 0 < y < 1$, find the corresponding pdf and use it to calculate $P(0 < X < \frac{1}{2}, \frac{1}{2} < Y < 1)$.

3.7.32. Prove that

$$P(a < X \le b, c < Y \le d) = F_{X,Y}(b, d) - F_{X,Y}(a, d) - F_{X,Y}(b, c) + F_{X,Y}(a, c)$$

3.7.33. A certain brand of fluorescent bulbs will last, on the average, 1000 hours. Suppose that four of these bulbs are installed in an office. What is probability that all four are still functioning after 1050 hours? If X_i denotes the ith bulb's life, assume that

$$f_{X_1,X_2,X_3,X_4}(x_1, x_2, x_3, x_4) = \prod_{i=1}^{4}\left(\frac{1}{1000}\right)e^{-x/1000}$$

for $x_i > 0, i = 1, 2, 3, 4$.

3.7.34. A hand of six cards is dealt from a standard poker deck. Let X denote the number of aces, Y the number of kings, and Z the number of queens.

(a) Write a formula for $p_{X,Y,Z}(x, y, z)$.
(b) Find $p_{X,Y}(x, y)$ and $p_{X,Z}(x, z)$.

3.7.35. Calculate $p_{X,Y}(0, 1)$ if $p_{X,Y,Z}(x, y, z) = \frac{3!}{x!y!z!(3-x-y-z)!}\left(\frac{1}{2}\right)^x\left(\frac{1}{12}\right)^y\left(\frac{1}{6}\right)^z \cdot \left(\frac{1}{4}\right)^{3-x-y-z}$ for $x, y, z = 0, 1, 2, 3$ and $0 \le x + y + z \le 3$.

3.7.36. Suppose that the random variables X, Y, and Z have the multivariate pdf

$$f_{X,Y,Z}(x, y, z) = (x + y)e^{-z}$$

for $0 < x < 1, 0 < y < 1$, and $z > 0$. Find (a) $f_{X,Y}(x, y)$, (b) $f_{Y,Z}(y, z)$, and (c) $f_Z(z)$.

3.7.37. The four random variables W, X, Y, and Z have the multivariate pdf

$$f_{W,X,Y,Z}(w, x, y, z) = 16wxyz$$

for $0 < w < 1, 0 < x < 1, 0 < y < 1$, and $0 < z < 1$. Find the marginal pdf, $f_{W,X}(w, x)$, and use it to compute $P(0 < W < \frac{1}{2}, \frac{1}{2} < X < 1)$.

Independence of Two Random Variables

The concept of independent events that was introduced in Section 2.5 leads quite naturally to a similar definition for independent random variables.

Definition 3.7.5. Two random variables X and Y are said to be *independent* if for every interval A and every interval B, $P(X \in A \text{ and } Y \in B) = P(X \in A)P(Y \in B)$.

Questions

3.7.38. Two fair dice are tossed. Let X denote the number appearing on the first die and Y the number on the second. Show that X and Y are independent.

3.7.39. Let $f_{X,Y}(x, y) = \lambda^2 e^{-\lambda(x+y)}$, $0 \le x$, $0 \le y$. Show that X and Y are independent. What are the marginal pdfs in this case?

3.7.40. Suppose that each of two urns has four chips, numbered 1 through 4. A chip is drawn from the first urn and bears the number X. That chip is added to the second urn. A chip is then drawn from the second urn. Call its number Y.

(a) Find $p_{X,Y}(x, y)$.
(b) Show that $p_X(k) = p_Y(k) = \frac{1}{4}$, $k = 1, 2, 3, 4$.
(c) Show that X and Y are not independent.

3.7.41. Let X and Y be random variables with joint pdf

$$f_{X,Y}(x, y) = k, \qquad 0 \le x \le 1, \quad 0 \le y \le 1, \quad 0 \le x + y \le 1$$

Give a geometric argument to show that X and Y are not independent.

3.7.42. Are the random variables X and Y independent if $f_{X,Y}(x, y) = \frac{2}{3}(x + 2y)$, $0 \le x \le 1$, $0 \le y \le 1$?

3.7.43. Suppose that random variables X and Y are independent with marginal pdfs $f_X(x) = 2x$, $0 \le x \le 1$, and $f_Y(y) = 3y^2$, $0 \le y \le 1$. Find $P(Y < X)$.

3.7.44. Find the joint cdf of the independent random variables X and Y, where $f_X(x) = \dfrac{x}{2}$, $0 \le x \le 2$, and $f_Y(y) = 2y$, $0 \le y \le 1$.

3.7.45. If two random variables X and Y are independent with marginal pdfs $f_X(x) = 2x$, $0 \le x \le 1$, and $f_Y(y) = 1$, $0 \le y \le 1$, calculate $P\left(\frac{Y}{X} > 2\right)$.

3.7.46. Suppose $f_{X,Y}(x, y) = xye^{-(x+y)}$, $x > 0$, $y > 0$. Prove for any real numbers a, b, c, and d that

$$P(a < X < b, c < Y < d) = P(a < X < b) \cdot P(c < Y < d)$$

thereby establishing the independence of X and Y.

3.7.47. Given the joint pdf $f_{X,Y}(x, y) = 2x + y - 2xy$, $0 < x < 1$, $0 < y < 1$, find numbers a, b, c, and d such that

$$P(a < X < b, c < Y < d) \ne P(a < X < b) \cdot P(c < Y < d)$$

thus demonstrating that X and Y are not independent.

3.7.48. Prove that if X and Y are two independent random variables, then $U = g(X)$ and $V = h(Y)$ are also independent.

3.7.49. If two random variables X and Y are defined over a region in the XY-plane that is *not* a rectangle (possibly infinite) with sides parallel to the coordinate axes, can X and Y be independent?

3.7.50. Write down the joint probability density function for a random sample of size n drawn from the exponential pdf, $f_X(x) = (1/\lambda)e^{-x/\lambda}$, $x \ge 0$.

3.7.51. Suppose that X_1, X_2, X_3, and X_4 are independent random variables, each with pdf $f_{X_i}(x_i) = 4x_i^3$, $0 \le x_i \le 1$. Find

(a) $P\left(X_1 < \frac{1}{2}\right)$.
(b) $P\left(\text{exactly one } X_i < \frac{1}{2}\right)$.
(c) $f_{X_1,X_2,X_3,X_4}(x_1, x_2, x_3, x_4)$.
(d) $F_{X_2,X_3}(x_2, x_3)$.

3.7.52. A random sample of size $n = 2k$ is taken from a uniform pdf defined over the unit interval. Calculate $P\left(X_1 < \frac{1}{2}, X_2 > \frac{1}{2}, X_3 < \frac{1}{2}, X_4 > \frac{1}{2}, \ldots, X_{2k} > \frac{1}{2}\right)$.

3.8 Transforming and Combining Random Variables

Transformations

Transforming a variable from one scale to another is a problem that is comfortably familiar. If a thermometer says the temperature outside is 83°F, we know that the temperature *in degrees Celsius* is *28*:

$$°\mathrm{C} = \left(\frac{5}{9}\right)(°\mathrm{F} - 32) = \left(\frac{5}{9}\right)(83 - 32) = 28$$

An analogous question arises in connection with random variables. Suppose that X is a discrete random variable with pdf $p_X(k)$. If a second random variable, Y, is defined to be $aX + b$, where a and b are constants, what can be said about the pdf for Y?