# Oppgåve 1 Løysingsforslag ST1101/6101

$X \sim N(\mu, \sigma^2)$

a) $P(X > 240) = P\left(\frac{X-\mu}{\sigma} > \frac{240-\mu}{\sigma}\right) = 1 - \Phi\left(\frac{240-\mu}{\sigma}\right)$

$\mu = 230,\ \sigma = 14$

$1 - \Phi\left(\frac{240-230}{14}\right) = 1 - \Phi(0.7143)$

$= 1 - 0.7625 = 0.2375$

$\mu = 225,\ \sigma = 15$

$1 - \Phi\left(\frac{240-225}{15}\right) = 1 - \Phi(1)$

$= 1 - 0.8413 = 0.1587$

$\mu = 227,\ \sigma = 13$

$1 - \Phi\left(\frac{240-227}{13}\right) = 1 - \Phi(1)$

$= 1 - 0.8413 = 0.1587$

$P(210 < X \leq 240) = P\left(\frac{210-\mu}{\sigma} < \frac{X-\mu}{\sigma} \leq \frac{240-\mu}{\sigma}\right) = \Phi\left(\frac{240-\mu}{\sigma}\right) - \Phi\left(\frac{210-\mu}{\sigma}\right)$

$\mu = 230,\ \sigma = 14$

$\Phi\left(\frac{240-230}{14}\right) - \Phi\left(\frac{210-230}{14}\right)$

$= \Phi(0.7143) - \Phi(-1.4286)$

$= 0.7625 - 0.0766 = 0.6859$

$\mu = 225,\ \sigma = 15$

$\Phi\left(\frac{240-225}{15}\right) - \Phi\left(\frac{210-225}{15}\right)$

$= \Phi(1) - \Phi(-1)$

$= 0.8413 - 0.1587 = 0.6826$

$\mu = 227,\ \sigma = 13$

$\Phi\left(\frac{240-227}{13}\right) - \Phi\left(\frac{210-227}{13}\right)$

$= \Phi(1) - \Phi(-1.3077)$

$= 0.8413 - 0.0954 = 0.7459$

b) $Y = \sum_{i=1}^{4} X_i \Rightarrow E[Y] = 4\mu,\ Var[Y] = 4\sigma^2 \Rightarrow SD(Y) = 2\sigma.$

$\mu = 230,\ \sigma = 14$

$P(Y > 960) = 1 - \Phi\left(\frac{960-920}{28}\right)$

$= 1 - \Phi(1.4286) = 1 - 0.9234$

$= \underline{0.0766}$

$\mu = 225,\ \sigma = 15$

$P(Y > 960) = 1 - \Phi\left(\frac{960-900}{30}\right)$

$= 1 - \Phi(2) = 1 - 0.9772$

$= \underline{0.0228}$

$\mu = 227,\ \sigma = 13$

$P(Y > 960) = 1 - \Phi\left(\frac{960-908}{26}\right)$

$= 1 - \Phi(2) = 0.0228$

$P(Y \leq k) = 0.95 \Leftrightarrow P\left(\frac{Y-4\mu}{2\sigma} \leq \frac{k-4\mu}{2\sigma}\right) = 0.95 \Leftrightarrow \Phi\left(\frac{k-4\mu}{2\sigma}\right) = 0.95$

$\Leftrightarrow k = 4\mu + 2\sigma \cdot 1.645$

$\mu = 230,\ \sigma = 14$

$k = 920 + 3.29 \cdot 14 = 966.06$

$\mu = 225,\ \sigma = 15$

$k = 900 + 3.29 \cdot 15 = 949.35$

$\mu = 227,\ \sigma = 13$

$k = 908 + 3.29 \cdot 13 = 950.77.$

## Oppgåve 2

a) $P(C) = P(C \cap G) + P(C \cap G^c) = P(C|G) \cdot P(G) + P(C|G^c) \cdot P(G^c)$

$= 1 \cdot 0.6 + P(C|G^c) \cdot 0.4$

$$P(G|C) = \frac{P(C \cap G)}{P(c)} = \frac{0.6}{P(C)}$$

$P(C|G^c) = 0.2$

$\Rightarrow P(C) = 0.6 + 0.2 \cdot 0.4 = 0.68$

$P(G|C) = \frac{0.6}{0.68} = 0,882$

$P(C|G^c) = 0,15$

$\Rightarrow P(C) = 0.6 + 0,15 \cdot 0.4 = 0.66$

$P(G|C) = \frac{0.6}{0,66} = 0.91$

$P(C|G^c) = 0.1$

$\Rightarrow P(C) = 0.6 + 0,1 \cdot 0.4 = 0,64$

$P(G|C) = \frac{0.6}{0.64} = 0.938$

## Oppgåve 3

a) $E[T] = \int_0^\infty t\lambda e^{-\lambda t}\,dt = \underset{0}{\overset{\parallel}{\left[-te^{-\lambda t}\right]_0^\infty}} + \int_0^\infty e^{-\lambda t}\,dt = \left[-\frac{1}{\lambda}e^{-\lambda t}\right]_0^\infty = \frac{1}{\lambda}$

La $m$ vere medianen til $T$. $\int_0^m \lambda e^{-\lambda t}\,dt = \frac{1}{2} \Rightarrow \left[-e^{-\lambda t}\right]_0^m = \frac{1}{2}$

$\Rightarrow 1 - e^{-\lambda m} = \frac{1}{2} \Rightarrow e^{-\lambda m} = \frac{1}{2} \Rightarrow m = \frac{\ln 2}{\lambda} = E[T]\cdot \ln 2 = 0{,}693 \cdot E[T]$

$P(T > 1.5\lambda) = \int_{1.5\lambda}^\infty \lambda e^{-\lambda t} = \left[-e^{-\lambda t}\right]_{1.5\lambda}^\infty = e^{-1.5\lambda^2}$

| $\lambda = 1$ | $\lambda = 0.8$ | $\lambda = 0{,}5$ |
|---|---|---|
| $E[T] = 1$ | $E[T] = 1.25$ | $E[T] = 2$ |
| $m = 0{,}693$ | $m = 0{,}866$ | $m = 1.386$ |
| $P(T > \overset{1.5}{1.5\lambda}) = e^{-1.5} = 0{,}\underline{223}$ | $P(T > \overset{1.2}{1.5\lambda}) = e^{-1.5\cdot(0{,}8)^2} = e^{-0{,}96} = \underline{0{,}383}$ | $P(T > \overset{0.75}{1.5\lambda}) = e^{-1.5\cdot(0{,}5)^2} = e^{-0{,}375} = \underline{0.687}$ |

b) $\underset{E[e^{sT}]}{\overset{M(s)}{\parallel}} = \int_0^\infty e^{st}\cdot\lambda e^{-\lambda t}\,dt = \lambda\int_0^\infty e^{(s-\lambda)t}\,dt \overset{s<\lambda}{=\!=} \lambda\left[\frac{e^{(s-\lambda)t}}{s-\lambda}\right]_0^\infty = \frac{\lambda}{\lambda - s}$ for $s < \lambda$

$\frac{d}{ds}M(s) = \frac{\lambda}{(\lambda - s)^2}$ , $\frac{d^2}{ds^2}M(s) = \frac{2\lambda}{(\lambda - s)^3}$

$M''(0) = \frac{2}{\lambda^2} \Rightarrow Var[X] = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$

| $\lambda = 1$ | $\lambda = 0.8$ | $\lambda = 0{,}5$ |
|---|---|---|
| $M(s) = \frac{1}{1-s}$ , $s < 1$ | $M(s) = \frac{0.8}{0.8 - s}$ , $s < 0{,}8$ | $M(s) = \frac{0{,}5}{0{,}5 - s}$ , $s < 0{,}5$ |
| $Var[T] = 1$ | $Var[T] = \frac{1}{(0{,}8)^2} = 1.563$ | $Var[T] = \frac{1}{(0{,}5)^2} = 4$ |

c) $Y = T_1 + T_2$ er ein sum av to uavhengige identisk ~~fordelte~~ eksponentialfordelte variable og dermed gammafordelt med parametrar 2 og $\lambda > 0$:

$$f_Y(y) = \begin{cases} \frac{\lambda^2}{\Gamma(2)}\, y e^{-\lambda y} & , y > 0 \\ 0 & , \text{elles} \end{cases} = \begin{cases} \lambda^2 y e^{-\lambda y} & , y > 0 \\ 0 & , \text{elles} \end{cases}$$

$P(Y \geq 3\lambda) = \int_{3\lambda}^\infty \lambda^2 y e^{-\lambda y}\,dy = \lambda^2\int_{3\lambda}^\infty y e^{-\lambda y}\,dy = \lambda^2\left[-y\cdot\frac{1}{\lambda}e^{-\lambda y}\right]_{3\lambda}^\infty + \lambda^2\int_{3\lambda}^\infty \frac{1}{\lambda}e^{-\lambda y}\,dy$

$= 3\lambda^2 e^{-3\lambda^2} + e^{-3\lambda^2} = \underline{e^{-3\lambda^2}[3\lambda^2 + 1]}$

$\lambda = 1$

$$f_Y(y) = \begin{cases} y e^{-y}, & y > 0 \\ 0, & \text{ellers} \end{cases}$$

$P(Y \geq 3) = 4 e^{-3} = 0.199$

$\lambda = 0,8$

$$f_Y(y) = \begin{cases} 0.64\, y e^{-0.8y}, & y > 0 \\ 0, & \text{ellers} \end{cases}$$

$P(Y \geq 2.4) = 2.92 e^{-1.92} = 0.428$

$\lambda = \frac{1}{2}$

$$f_Y(y) = \begin{cases} 0.25 \cdot y e^{-0,5y}, & y > 0 \\ 0, & \text{ellers}. \end{cases}$$

$P(Y \geq 1,5) = 1.75 e^{-0,75} = 0,827$

d) $P(\min(T_1, T_2) < 0.5\lambda) = 1 - P(T_1 > 0,5\lambda) \cdot P(T_2 > 0,5\lambda)$

$= 1 - e^{-0,5\lambda^2} \cdot e^{-0,5\lambda^2} = 1 - e^{-\lambda^2}$

$\lambda = 1$

$P(\min(T_1, T_2) < 0,5) = 1 - e^{-1}$

$= 0,632$

$\lambda = 0,8$

$P(\min(T_1, T_2) < 0,4) = 1 - e^{-0,64}$

$= 0,473$

$\lambda = 0,5$

$P(\min(T_1, T_2) < 0,25) = 1 - e^{-0,25}$

$0,221$

$$P\left(T_2 < \frac{T_1}{2}\right) = \int_0^\infty \lambda e^{-\lambda t_1} \left( \int_0^{\frac{t_1}{2}} \lambda e^{-\lambda t_2}\, dt_2 \right) dt_1$$

$$= \int_0^\infty \lambda e^{-\lambda t_1} \left(1 - e^{-\lambda \frac{t_1}{2}}\right) dt_1 = \int_0^\infty \lambda e^{-\lambda t_1} - \int_0^\infty \lambda e^{-\frac{3}{2}\lambda t_1}\, dt_1$$

$$= 1 - \left[ -\frac{2}{3} e^{-\frac{3}{2}\lambda t_1} \right]_0^\infty = 1 - \frac{2}{3} = \frac{1}{3}$$

# Oppgåve 4a

Summen av uavhengige poissonfordelte variable er poissonfordelt med parameter summen av parametrane for dei enkelte.

Difor er $Y = \sum_{i=1}^{20} X_i \sim Po(\sum_{i=1}^{20} \lambda) \sim Po(20\lambda)$

$k > 3 \Rightarrow P(Y > k \mid X_1 + X_2 = 3) = \dfrac{P(Y > k \cap \sum_{i=1}^{2} X_i = 3)}{P(\sum_{i=1}^{2} X_i = 3)}$

$= \dfrac{P(\sum_{i=3}^{20} X_i > k-3 \cap \sum_{i=1}^{2} X_i = 3)}{P(\sum_{i=1}^{2} X_i = 3)} = \dfrac{(1 - P(\sum_{i=3}^{20} X_i \le k-3)) \cdot P(\sum_{i=1}^{2} X_i = 3)}{P(\sum_{i=1}^{2} X_i = 3)}$

$= 1 - P(\sum_{i=3}^{20} X_i \le k-3)$, $\quad \sum_{i=3}^{20} X_i \sim Po(18\lambda)$

$\lambda = \frac{1}{2} \Rightarrow Y \sim Po(10)$ og $\sum_{i=3}^{20} X_i \sim Po(9)$

**$k = 10$**

$P(Y > 10) = 1 - P(Y \le 10)$

$= 1 - 0.583 = 0.417$

$P(Y > 10 \mid \sum_{i=1}^{2} X_i = 3) = 1 - P(\sum_{i=3}^{20} X_i \le 7)$

$= 1 - 0.3239 = 0.6761$

**$k = 11$**

$P(Y > 11) = 1 - P(Y \le 11)$

$= 1 - 0.6968 = 0.3032$

$P(Y > 11 \mid \sum_{i=1}^{2} X_i = 3) = 1 - P(\sum_{i=3}^{20} X_i \le 8)$

$= 1 - 0.4557 = \underline{0.5443}$

**$k = 12$**

$P(Y > 12) = 1 - P(Y \le 12)$

$= 1 - 0.7916 = 0.2084$

$P(Y > 12 \mid \sum_{i=1}^{2} X_i = 3) = 1 - P(\sum_{i=3}^{20} X_i \le 9)$

$1 - 0.5874 = \underline{0.4126}$

b). $H_0 : \lambda = \frac{1}{2}$, $\quad H_1 : \lambda > \frac{1}{2}$

$\alpha = 0.05 \Rightarrow$ forkast $H_0$ når $Y \ge k$ der $P(Y \ge k) \le \alpha$

$P(Y \ge k \mid \lambda = \frac{1}{2}) = 1 - P(Y \le k-1 \mid \lambda = \frac{1}{2})$

$$\left.\begin{array}{l} k-1 = 14 \Rightarrow P(Y \ge k) = 0.0835 \\ k-1 = 15 \Rightarrow P(Y \ge k) = 0.0487 \end{array}\right\} \Rightarrow \text{Forkast når } Y \ge 16$$

Observert Y er 14

Ikkje forkast på nivå 0,05

Observert Y = 16

Forkast på nivå 0,05

Observert Y = 18

Forkast på nivå 0,005.

c)

$$P(Y=y \mid N=m) = \frac{P(Y=y \cap N=m)}{P(N=m)} = \frac{P(Y=y \cap \sum_{i=1}^{m} Z_i = m-y)}{P(N=m)}$$

$$= \frac{\frac{(20\lambda)^y e^{-20\lambda}}{y!} \cdot \frac{(40\lambda)^{m-y} e^{-40\lambda}}{(m-y)!}}{\frac{(60\lambda)^m e^{-60\lambda}}{m!}} = \frac{m!}{y!(m-y)!} \frac{20^y \lambda^y \, 40^{m-y} \lambda^{m-y}}{60^y \, 60^{m-y} \lambda^m}$$

$$= \binom{m}{y}\left(\frac{1}{3}\right)^y\left(1-\frac{1}{3}\right)^{m-y} \quad \partial; \quad B(m, \tfrac{1}{3})$$

$$E[Y \mid N=m] = \frac{m}{3}, \quad Var[Y \mid N=m] = \frac{m}{3} \cdot \frac{2}{3} = \frac{2m}{9}$$

$m = 30$

$$P(Y \geq 16 \mid N=30) \approx P\left(\frac{Y-10}{\sqrt{\frac{20}{3}}} \geq \frac{15{,}5-10}{\sqrt{\frac{20}{3}}}\right)$$

$$\approx P\left(Z \geq \frac{5{,}5}{\sqrt{\frac{20}{3}}}\right) = P(Z \geq 2{,}13)$$

$$= P(Z \leq -2{,}13) = 0.017$$

$m = 27$

$$P(Y \geq 16 \mid N=27) \approx P\left(\frac{Y-9}{\sqrt{6}} \geq \frac{15{,}5-9}{\sqrt{6}}\right)$$

$$\approx P\left(Z \geq \frac{6{,}5}{\sqrt{6}}\right) = P(Z \geq 2{,}65)$$

$$= P(Z \leq -2{,}65) = 0.004$$

$m = 24,$

$$P(Y \geq 16 \mid N=24) \approx P\left(\frac{Y-8}{\sqrt{\frac{16}{3}}} \geq \frac{15{,}5-8}{\sqrt{\frac{16}{3}}}\right)$$

$$\approx P\left(Z \geq \frac{7{,}5}{\sqrt{\frac{16}{3}}}\right) = P(Z \geq 3{,}25)$$

$$= P(Z \leq -3{,}25) = 0.0006$$