Løsningsforslag (ST1101/ST6101 vår 2014, kontinuasjonseksamen)

1.

a) la S være hendelsen at individet er sykt og la aa, Aa og AA være hendelsene at individet er av hver av de tre respektive genotypene. Lov om totalsannsynlighet gir da

$$P(S) = P(S|aa)P(aa) + P(S|Aa)P(Aa) + P(S|AA)P(AA) =$$

$$= 0.6 \cdot 0.0001 + 0.0198 \cdot 0.02 + 0.9801 \cdot 0.01 = 0.010257.$$

b) Den betingede sannsynligheten for at et sykt individ er av type aa blir i følge Bayes teorem

$$P(aa|S) = \frac{P(S|aa)P(aa)}{P(S)} = \frac{0.6 \cdot 0.0001}{0.010257} = 0.00584.$$

På tilsvarende måte får vi

$$P(Aa|S) = 0.0386$$

og

$$P(AA|S) = 0.956.$$

2.

a) For $t \geq 0$,

$$F_T(t) = \int_{-\infty}^{t} f_T(x)dx = \int_0^t 2\lambda x e^{-\lambda x^2} dx = 1 - e^{-\lambda t^2}.$$

b)

$$P(T > 2/\sqrt{\lambda} | T > 1/\sqrt{\lambda}) = \frac{P(T > 2/\sqrt{\lambda}, T > 1/\sqrt{\lambda})}{P(T > 1/\sqrt{\lambda})} =$$

$$= \frac{P(T > 2/\sqrt{\lambda})}{P(T > 1/\sqrt{\lambda})} = \frac{1 - F_T(2/\sqrt{\lambda})}{1 - F_T(1/\sqrt{\lambda})} = \frac{e^{-\lambda(2/\sqrt{\lambda})^2}}{e^{-\lambda(2/\sqrt{\lambda})^2}} = \frac{e^{-4}}{e^{-1}} = e^{-3}.$$

c) Rimelighetsfunksjonen er

$$L(T_1, ..., T_n) = \prod_{i=1}^{n} f_T(T_i) = 2^n \lambda^n (\prod_{i=1}^{n} T_i) e^{-\lambda \sum_{i=1}^{n} T_i^2}.$$

Logaritme

$$\ln L = n \ln \lambda + \ln(2^n \prod_{i=1}^{n} T_i) - \lambda \sum_{i=1}^{n} T_i^2.$$

Bestemmer makspunktet $\hat{\lambda}_{\text{SME}}$ ved å løse ligningen

$$\frac{d\ln L}{d\lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} T_i^2 = 0.$$

Løsningen er

$$\hat{\lambda}_{\text{SME}} = \frac{n}{\sum_{i=1}^{n} T_i^2}.$$

3.

a) Denote $A = \{(x,y): \ y < x\}$. Then

$$P(Y < X) = P((X,Y) \in A) = \int\int_A f_{X,Y}(x,y)dxdy =$$

$$= \int_0^1 \left(\int_0^x f_{X,Y}(x,y)dy\right) dx = \int_0^1 \left(\int_0^x (x+y)dy\right) dx =$$

$$= \int_0^1 \frac{3x^2}{2} dx = \frac{1}{2}.$$

b)

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx = \int_0^1 (x+y)dx = y + \frac{1}{2}.$$

We have

$$P(Y < X | Y < 1/2) = \frac{P(Y < X, Y < 1/2)}{P(Y < 1/2)}.$$

Denote $B = \{(x,y): \ y < x, y < 1/2\}$. Then

$$P(Y < X, Y < 1/2) = P((X,Y) \in B) = \int\int_B f_{X,Y} dxdy =$$

$$= \int_0^{1/2} \left(\int_0^x f_{X,Y}(x,y)dy\right) dx + \int_{1/2}^1 \left(\int_0^{1/2} f_{X,Y}(x,y)dy\right) dx =$$

$$= \int_0^{1/2} \frac{3x^2}{2} dx + \int_{1/2}^1 \left(\frac{x}{2} + \frac{1}{8}\right) dx = \frac{5}{16},$$

$$P(Y < 1/2) = \int_0^{1/2} f_Y(y)dy = \int_0^{1/2} \left(y + \frac{1}{2}\right) dy = \frac{3}{8},$$

and thus

$$P(Y < X | Y < 1/2) = \frac{5/16}{3/8} = \frac{5}{6}.$$

4.

a) Yes. Consider the following example. Let a die be tossed one time. Consider

$$A = \{1,2\}, \ B = \{1,3\}, \ C = \{1,4,5\}.$$

Then

$$P(A) = 1/3,\ P(B) = 1/3,\ P(C) = 1/2.$$

$$P(A \cap C) = 1/6 = P(A)P(C)$$

$$P(B \cap C) = 1/6 = P(B)P(C)$$

$$P((A \cap B) \cap C) = 1/6 \neq 1/12 = P(A \cap B)P(C)$$

$$P((A \cup B) \cap C) = 1/6 \neq 1/4 = P(A \cup B)P(C)$$

b) Now they cannot be dependent. If A and B are disjoint, then

$$P((A \cap B) \cap C) = 0 = P(A \cap B)P(C)$$

i.e. $A \cap B$ and C are always independent.

$$P((A \cup B) \cap C) = P((A \cap C) \cup (B \cap C) = P(A \cap C) + P(B \cap C) =$$

$$= P(A)P(C) + P(B)P(C) = P(C)(P(A) + P(B)) = P(A \cup B)P(C)$$

i.e. $A \cup B$ and C are always independent.

5.

a) The first equality implies that $VarX = (EX)^2$ i.e.

$$\frac{r}{\lambda^2} = \frac{r^2}{\lambda^2},$$

and therefore $r = 1$. Now from the secon equality we obtain

$$Ee^X = M_X(1) = \frac{\lambda}{\lambda - 1} = 2$$

i.e. $\lambda = 2$.