Løsningsforslag (ST1101/ST6101 vår 2014)

1. A coin is tossed two times. Consider the following events $A = \{\text{the head in the first toss}\}$, $B = \{\text{the head in the second toss}\}$, $C = \{\text{one head and one tail}\}$.

a) Are A, B, C pairwise independent? Are A, B, C independent? Explain you answers.

Solution. Possible outcomes: hh, ht, th, tt. Then

$$A = \{\text{hh}, \text{ht}\}, \ P(A) = \frac{1}{2}$$

$$B = \{\text{hh}, \text{th}\}, \ P(B) = \frac{1}{2}$$

$$C = \{\text{ht}, \text{th}\}, \ P(C) = \frac{1}{2}$$

$$A \cap B = \{\text{hh}\}, \ P(A \cap B) = \frac{1}{4} = P(A)P(B)$$

$$A \cap C = \{\text{ht}\}, \ P(A \cap C) = \frac{1}{4} = P(A)P(C)$$

$$B \cap C = \{\text{th}\}, \ P(B \cap C) = \frac{1}{4} = P(B)P(C)$$

Events are pairwise independent.

$$A \cap B \cap C = \emptyset, \ P(A \cap B \cap C) = 0 \neq P(A)P(B)P(C).$$

Evets are not independent.

2. Let X be a continuous random variable with the probability density function

$$f_X(x) = \begin{cases} \frac{1}{3}(4x+1) & \text{for } x \in [0,1], \\ 0 & \text{otherwise,} \end{cases}$$

and let Y be a continuous random variable with the conditional density (given $X = x$)

$$f_{Y|X=x}(y) = \begin{cases} \frac{4x+2y}{4x+1} & \text{for } y \in [0,1], \\ 0 & \text{otherwise.} \end{cases}$$

a) Find $P(Y \leq X)$.
b) Find $P(X \leq 1/2 | Y \leq 1/2)$.
c) Are variables X and Y independent?
Solution. Let us find the joint density of X and Y:

$$f_{X,Y}(x,y) = f_{Y|X=x}(y) f_X(x) = \frac{2}{3}(2x + y)$$

for $0 \leq x \leq 1$, $0 \leq y \leq 1$ (otherwise 0).
a) Denote $A = \{(x,y) : \ y \leq x\}$. Then

$$P(Y \leq X) = P((X,Y) \in A) = \int \int_A f_{X,Y}(x,y) dx dy =$$

$$= \int_0^1 \left(\int_0^x \frac{2}{3}(2x + y) dy \right) dx = \frac{5}{9}.$$

b)

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) dx dy = \int_0^1 \frac{2}{3}(2x + y) dx = \frac{2}{3}(y + 1).$$

$$P(Y \leq 1/2) = \int_{-\infty}^{1/2} f_y(y) dy = \int_0^{1/2} \frac{2}{3}(y + 1) dy = \frac{5}{12}.$$

$$P(X \leq 1/2, Y \leq 1/2) = \int_0^{1/2} \left(\int_0^{1/2} \frac{2}{3}(2x + y) dy \right) dx = \frac{1}{8}.$$

Thus

$$P(X \leq 1/2 | Y \leq 1/2) = \frac{P(X \leq 1/2, Y \leq 1/2)}{(Y \leq 1/2)} = \frac{3}{10}.$$

c) No: $f_{X,Y}(x,y) \neq f_X(x) f_Y(y)$.

3. Y has the normal distribution with the expectation μ and variance σ^2.
a) Find the expectation and the variance of the random variable e^Y.

b) Find $P(\mu - 2\sigma < Y < \mu + 2\sigma)$.

c) Let $P(Y < 0) = 0.5$ and $P(Y < 1) = 0.6915$. What are μ and σ^2?

Solution. a) The moment generating function of Y is

$$M_Y(t) = e^{\mu t + \sigma^2 t^2/2}$$

therefore

$$Ee^Y = M_Y(1) = e^{\mu + \sigma^2/2}.$$

The variance

$$Var(e^Y) = E(e^Y)^2 - (Ee^Y)^2 =$$
$$= M_Y(2) - (M_Y(1))^2 = e^{2\mu+\sigma^2}(e^{\sigma^2} - 1).$$

b)

$$P(\mu - 2\sigma < Y < \mu + 2\sigma) = P\left(-2 < \frac{Y-\mu}{\sigma} < 2\right) =$$
$$= P(-2 < Z < 2) = \Phi(2) - \Phi(-2) = 2\Phi(2) - 1 = 0.9544.$$

c) The first equality implies that $\mu = 0$. Then

$$P(Y < 1) = P\left(\frac{Y}{\sigma} < \frac{1}{\sigma}\right) = P\left(Z < \frac{1}{\sigma}\right) = 0.6915$$

that implies that $1/\sigma = 0.5$ i.e. $\sigma^2 = 4$.

4. Two neighbouring sides in a rectangle are independent random variables X and Y with the same exponential distribution with parameter θ ($EX = EY = 1/\theta$).

a) Show that the distribution of the perimeter T of the rectangle has the gamma distribution with parameters 2 and $\theta/2$.

Suppose that θ is unknown. It is estimated on the basis of a sample of n independent circumferences $T_1, ..., T_n$, where the distribution of T_i, $i = 1, ..., n$, is the same as in item a).

b) Find the maximum likelihood estimator and the method of moment estimator.

Solution. a) Denote density functions of X (and Y) and T by $f(x)$ and $f_T(t)$, respectively. Then

$$f(x) = \theta e^{-\theta x}, \ x \geq 0,$$

and therefore the density of $X+Y$ is

$$f_{X+Y}(t)=\int_{-\infty}^{\infty} f(t-x)f(x)dx=\int_{\theta}^{t} \theta e^{-\theta(t-x)}\theta e^{-\theta x}dx=\theta^2 te^{-\theta t}.$$

The density of T is

$$f_T(t)=\frac{1}{2}f_{X+Y}(t/2)=\frac{\theta^2}{4}te^{-\theta t/2}.$$

This is the density of the gamma distribution with parameters $(2,\theta/2)$.

b) The likelihood function is

$$L(\theta;X_1,...,T_n)=\prod_{i=1}^{n}\frac{\theta^2}{4}T_i e^{-\theta T_i/2}=\frac{\theta^{2n}}{4^n}\left(\prod_{i=1}^{n}T_i\right)e^{-\frac{\theta}{2}\sum T_i}.$$

Loglikelihood is

$$\ln L=2n\ln\theta-n\ln 4+\ln\prod_{i=1}^{n}T_i-\frac{\theta}{2}\sum_{i=1}^{n}T_i.$$

The derivative

$$\frac{\partial \ln L}{\partial\theta}=\frac{2n}{\theta}-\frac{1}{2}\sum_{i=1}^{n}T_i,$$

and the maximum likelihood estimator (solution of the likelihood equation) is

$$\hat{\theta}_{\text{MLE}}=\frac{4n}{\sum_{i=1}^{n}T_i}.$$

The moment estimator is the solution of the equation

$$\frac{m}{\theta}=M_1$$

where M_1 is the first empirical moment i.e.

$$M_1=\frac{1}{n}\sum_{i=1}^{n}T_i.$$

This solution coincides with the maximum likelihood estimator.

5. A random variable N has the binomial distribution with parameters n and p i.e.

$$p_N(k)=P(N=k)=\binom{n}{k}p^k(1-p)^{n-k},\ k=0,1,...,n.$$

Let $Y_0, Y_1, ..., Y_n$ be continuous random variables , and let Y_k and N be independent for all $k = 0, 1, ..., n$. Denote $X = Y_N$.

Suppose that

$$EY_k = \mu^k \ (\mu \neq 0), \ k = 0, 1, ..., n.$$

Prove that

$$EX = (p\mu + 1 - p)^n.$$

Solution. Denote the cumulative distribution function and the density function of Y_k by $F_k(x)$, $f_k(x)$, and those of X by $F(x)$, $f(x)$, respectively. Then

$$F(x) = P(X \leq x) = \sum_{k=0}^{n} P(X \leq x | N = k) P(N = k) =$$

$$= \sum_{k=0}^{n} P(Y_N \leq x | N = k) P(N = k) =$$

$$= \sum_{k=0}^{n} P(Y_k \leq x | N = k) P(N = k) =$$

$$= \sum_{k=0}^{n} P(Y_k \leq x) P(N = k) = \sum_{k=0}^{n} p_N(k) F_k(x),$$

and therefore

$$f(x) = \sum_{k=0}^{n} p_N(k) f_k(x).$$

Now

$$EX = \int_{-\infty}^{\infty} x f(x) dx = \sum_{k=0}^{n} p_N(k) \int_{-\infty}^{\infty} x f_k(x) dx =$$

$$= \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} \mu^k = (p\mu + (1-p))^n.$$