Løsningsforslag (ST1201/ST6201 2017, kontinuasjonseksamen)

1.

a) The power function (probability to reject H_0) is

$$1 - \beta(\mu) = \Phi\left(\frac{\sqrt{n}}{\sigma}\mu - z_\alpha\right).$$

In our case

$$1 - \beta(1) = \Phi(2 - z_\alpha) = 0.5$$

i.e. $z_\alpha = 2$, $\alpha = 0.0228$.

b)

$$1 - \beta(2) = \Phi(4 - z_\alpha) = \Phi(2) = 0.9772.$$

2.

a) Under H_0 the test statistic of the test

$$\frac{\bar{X} - \bar{Y}}{\sqrt{\sigma_X^2/n + \sigma_Y^2/m}}$$

has the standard normal distribution, therefore the $P-$value is

$$p(x,y) = P_{\mu_X=\mu_Y}\left(\left|\frac{\bar{X} - \bar{Y}}{\sqrt{\sigma_X^2/n + \sigma_Y^2/m}}\right| \geq \frac{\bar{x} - \bar{y}}{\sqrt{\sigma_X^2/n + \sigma_Y^2/m}}\right) =$$

$$= 2\Phi(-1) \approx 0.32.$$

3.

a) Let X be the number of observations greater than 0.5. The large-sample sign test: H_0 is rejected if

$$Z = \frac{X - n/2}{\sqrt{n/4}} \geq z_\alpha.$$

In our case ($X = 11$, $n = 16$)

$$Z = 1.5 < 1.645 = z_\alpha$$

therefore H_0 is not rejected.

b) The large-sample Wilcoxon signed rank test: H_0 is rejected if

$$z = \frac{w - n(n+1)/4}{\sqrt{n(n+1)(2n+1)/24}} \geq z_\alpha,$$

where $w = \sum_{i=1}^{n} r_i z_i$, r_i – rank of $|y_i - 0.5|$, $z_i = 1$ if $y_i > 0.5$ and 0 otherwise.
In our case

i	1	2	3	4	5	6	7	8
$\|y_i - 0.5\|$	0.07	0.34	0.11	0.11	0.08	0.21	0.22	0.18
r_i	5	16	7.5	7.5	6	12	13.5	10
z_i	1	1	1	0	0	1	0	0
i	9	10	11	12	13	14	15	16
$\|y_i - 0.5\|$	0.13	0.01	0.02	0.22	0.19	0.27	0.03	0.06
r_i	9	1	2	13.5	11	15	3	4
z_i	1	1	0	1	1	1	1	1

$$w = 97$$

and

$$z = 1.5 < 1.645 = z_\alpha.$$

Thus H_0 is not rejected.

4.

a) Poisson probabilities ($\lambda = 1$) are

$$p_k = \frac{e^{-1}}{k!}, \ k = 0, 1, ...,$$

i.e.
$p_0 = 0.37$
$p_1 = 0.37$
$p_2 = 0.18$
$p_3 = 0.06$
$p_4 = 0.015$
$p_{5+} = 0.005$

To apply the Pearson goodness-of-fit test, we use the partition (of nonnegative integers)

$$\{0\}, \ \{1\}, \ \{2, 3, ...\}$$

with probabilities (under H_0)

$$p_{10} = 0.37, \ p_{20} = 0.37, \ p_{30} = 0.26.$$

This partition is chosen because the condition $np_{i0} \geq 5$ must be satisfied. Then

$$k_1 = 2, \ k_2 = 6, \ k_3 = 23;$$

$$np_{10} = 11.47,\ np_{20} = 11.47,\ np_{30} = 8.06.$$

The observed value of the test statistic is

$$d = \sum_{i=1}^{3} \frac{(k_i - np_{i0})^2}{np_{i0}} = 38.1 > 5.991 = \chi^2_{0.95,2}.$$

The hypothesis is rejected.

b) Maximum likelihood estimate of λ is $\hat{\lambda} = 2.48... \approx 2.5$ and estimates of Poisson probabilities are

$$\hat{p}_k = \frac{e^{-2.5}2.5^k}{k!},\ k = 0, 1, ...,$$

i.e.
$\hat{p}_0 = 0.08$
$\hat{p}_1 = 0.21$
$\hat{p}_2 = 0.26$
$\hat{p}_3 = 0.21$
$\hat{p}_4 = 0.13$
$\hat{p}_5 = 0.07$
$\hat{p}_{6+} = 0.04$

To apply the Pearson goodness-of-fit test, we use the partition (of nonnegative integers)

$$\{0, 1\},\ \{2\},\ \{3\},\ \{4, 5, ...\}$$

with probabilities (under H_0)

$$\hat{p}_{10} = 0.29,\ \hat{p}_{20} = 0.26,\ \hat{p}_{30} = 0.21,\ \hat{p}_{40} = 0.24.$$

This partition is chosen because the condition $n\hat{p}_{i0} \geq 5$ must be satisfied. Then

$$k_1 = 8,\ k_2 = 10,\ k_3 = 6,\ k_4 = 7;$$

$$n\hat{p}_{10} = 8.99,\ n\hat{p}_{20} = 8.06,\ n\hat{p}_{30} = 6.51,\ n\hat{p}_{40} = 7.44.$$

The observed value of the test statistic is

$$d_1 = \sum_{i=1}^{4} \frac{(k_i - n\hat{p}_{i0})^2}{n\hat{p}_{i0}} = 0.65 < 5.991 = \chi^2_{0.95,2}.$$

The hypothesis is accepted.

5.

a)

$$MSTR = F \cdot MSE = 1.6 \cdot 5.1 = 8.16$$

$$SSTR = MSTR \cdot df = 8.16 \cdot 3 = 24.48$$

$$SSE = MSE \cdot df = 5.1 \cdot 40 = 204$$

$$SSTOT = SSTR + SSE = 204 + 24.48 = 228.48$$

Thus the filled ANOVA table is

Source	df	SS	MS	F
Treatment	3	24.48	8.16	1.6
Error	40	204	5.1	
Total	43	228.48		

b) $F_{0.95,3,40} = 2.84 > 1.6$ (observed F). Therefore H_0 is not rejected.

6.

a) Denote probabilities

$$p_{11} = P(\text{good husband, good wife}),$$

$$p_{12} = P(\text{good husband, bad wife}),$$

$$p_{21} = P(\text{bad husband, good wife}),$$

$$p_{22} = P(\text{bad husband, bad wife}),$$

$$p_1 = P(\text{good husband}),$$

$$p_2 = P(\text{bad husband}),$$

$$q_1 = P(\text{good wife}),$$

$$q_2 = P(\text{bad wife}).$$

(It is clear that $p_1 + p_2 = 1$, $q_1 + q_2 = 1$, $p_{11} + p_{11} + p_{21} + p_{22} = 1$.) It is tested the null hypothesis $H_0 : p_{ij} = p_i q_j, i = 1,2, j = 1,2$. We are given the contingency table

X_{11}	X_{12}
X_{21}	X_{22}

The test statistic (we use notations of the textbook) is

$$D_2 = \sum_{i=1}^{2} \sum_{j=1}^{2} \frac{(X_{ij} - n\hat{p}_i\hat{q}_j)^2}{n\hat{p}_i\hat{q}_j}$$

where $\hat{p}_i, \hat{q}_j$ are estimators (MLE) of p_i, q_j. H_0 is rejected if

$$D_2 \geq \chi^2_{1-\alpha,1}.$$

In our case $D_2 = 1.02$, $\chi^2_{1-\alpha,1} = 3.841$. The null hypothesis is not rejected (that means that there is no reason to believe that the temperament of husband and wife are dependent of each other).