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English

Contact during exam:
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EXAM IN ST1201/ST6201 STATISTICAL METHODS

Friday December 8th 2006
Time: 09:00–13:00

Aids: All typed and written.
All calculators allowed.

Grading: December 29th 2006.

Oppgave 1

Let $X \sim \mathrm{N}(\mu, \sigma^2)$ and $Y \sim \mathrm{N}(\mu, 2\sigma^2)$ and assume X and Y to be independent. We want to estimate μ from observed values for X and Y. The following to estimators have been proposed,

$$\widehat{\mu} = \frac{X + \frac{1}{2}Y}{1 + \frac{1}{2}} \quad , \quad \tilde{\mu} = \frac{X + \frac{1}{\sqrt{2}}Y}{1 + \frac{1}{\sqrt{2}}}.$$

Which of the two estimators would you prefer? Give reason for the answer!

Oppgave 2

Let $X_1, X_2, \ldots, X_n$ be a random sample from a continuous distribution with probability density function

$$f(x) = \frac{xe^{-x/\beta}}{\beta^2} \quad , \quad x > 0.$$

It is given that $\mathrm{E}[X] = 2\beta$ and $\mathrm{Var}[X] = 2\beta^2$. The value of the parameter β is unknown and should be estimated.

a) Explain briefly how the moment estimators are defined in general (when the model has r unknown parameters $\theta_1, \theta_2, \ldots, \theta_r$).

Show that the moment estimator for the parameter β in the distribution above is given by

$$\widehat{\beta} = \frac{1}{2n} \sum_{i=1}^{n} X_i.$$

Let $Z_i = 2X_i/\beta$.

b) Show that Z_i is χ^2 distributed with four degrees of freedom.

Show/explain why $4n\widehat{\beta}/\beta \sim \chi^2_{4n}$. In particular specify known properties for the χ^2 distribution you are using to make this conclusion.

We want to use observed values for $X_1, X_2, \ldots, X_n$ to test whether it is reasonable to claim that $\beta > \beta_0$, where β_0 is a given number.

c) Write down H_0 and H_1 for this situation. Chose a test statistic and work out the corresponding decision rule with level of significance equal to α.

What is the conclusion of the test if $\beta_0 = 2$, $n = 10$, $\sum_{i=1}^{n} x_i = 25.87$ and $\alpha = 0.05$.

d) Work out an expression for the power curve for the test statistic found in the previous item (find an expression for general β_0, n and α).

For $\beta_0 = 2$, $n = 10$ and $\alpha = 0.05$, for what value of β is the power equal to 0.99? Give also a precise description of the event that has probability 0.99 here.

Oppgave 3

In this exercise we will consider a regression model that is somewhat modified relative to what is discussed in the textbook. Assume we have pairs of variables $(x_1, Y_1), (x_2, Y_2), \ldots, (x_n, Y_n)$ where $x_1, x_2, \ldots, x_n$ are considered not stochastic, whereas $Y_1, Y_2, \ldots, Y_n$ are assumed to be independent stochastic variables with

$$Y_i \sim \mathrm{N}(\alpha + \beta x_i, \sigma_0^2 x_i).$$

Thus, the variance of Y_i is assumed to be proportional to x_i. In this exercise we will assume the parameter value σ_0^2 to be known, whereas the values of the two parameters α and β are to be estimated from the available data.

a) Work out the maximum likelihood estimator (MLE) for α and β and show that they can be written on the form

$$\widehat{\alpha} = \bar{Y} - \widehat{\beta}\bar{x} \quad \text{and} \quad \widehat{\beta} = \frac{\bar{Y}\sum_{i=1}^{n}\frac{1}{x_i} - \sum_{i=1}^{n}\frac{Y_i}{x_i}}{\bar{x}\sum_{i=1}^{n}\frac{1}{x_i} - n},$$

where $\bar{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$ and $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$.

b) Show that $\widehat{\beta}$ is unbiased and show that

$$\mathrm{Var}[\widehat{\beta}] = \frac{\sigma_0^2}{n} \cdot \frac{\frac{1}{n}\sum_{i=1}^{n}\frac{1}{x_i}}{\bar{x}\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{x_i}\right) - 1}.$$

c) What is the probability distribution of $\widehat{\beta}$? Give reason for the answer, and specify in particular any known properties you are using to make the conclusion, and explain why these can be used in this situation.

Work out a $(1 - a) \cdot 100\%$ confidence interval for β.