Oppgave 1

a)

Man benyttet randomisert blokkdesign.
I situasjonen har man $k = 5,\ \ b = 15,\ \ n = kb = 75$.
ANOVA-tabellen blir

Kilde	df	SS	MS	F	P-verdi
Instrument	$k-1=4$	0.20219	$\frac{0.20219}{4} = 0.0505$	$\frac{0.0505}{0.004818} = 10.48$	0
Objekt	$b-1=14$	45.48159	$\frac{45.48159}{14} = 3.2487$	$\frac{3.2487}{0.004818} = 674.3$	0
Error	$74-4-14=56$	0.26981	$\frac{0.26981}{56} = 0.004818$		
Total	$n-1=74$	45.95359			

der

$$SSTR = 45.95359 - 0.26981 - 45.48159 = 0.20219.$$

Modellen er:

$$Y_{ij} = \mu_j + \beta_i + \epsilon_{ij}; \quad \epsilon_{ij} \sim N(0, \sigma^2),$$

hvor
μ_j: effekten av instrument nr. j
β_i: effekten av objekt nr. i
ϵ_{ij}: tilfeldig feil

b)

H_0 for p-verdi i første linje:

$$H_0: \quad \mu_1 = \mu_2 = \ldots = \mu_5$$

H_0 for p-verdi i andre linje:

$$H_0: \quad \beta_1 = \beta_2 = \ldots = \beta_{15}$$

Forskningsinstitusjonen er interessert i den første p-verdien, dvs. er det forskjell mellom instrumentene.

P-verdien er (avrundet til) null, så konklusjon blir **forkast** H_0, dvs. det er forskjell mellom måleinstrumentene.

Oppgave 2

Teoretiske momenter er

$$\mathrm{E}[X] = \frac{r}{\lambda}$$

$$\mathrm{E}[X^2] = \mathrm{Var}[X] + \big(\mathrm{E}[X]\big)^2 = \frac{r}{\lambda^2} + \left(\frac{r}{\lambda}\right)^2 = \frac{r(1+r)}{\lambda^2}$$

Momentestimatorene er dermed gitt ved

$$\frac{1}{n}\sum_{i=1}^{n} X_i = \frac{\widehat{r}}{\widehat{\lambda}}, \quad \frac{1}{n}\sum_{i=1}^{n} X_i^2 = \frac{\widehat{r}(1+\widehat{r})}{\widehat{\lambda}^2},$$

Dette gir

$$\widehat{r} = \frac{\widehat{\lambda}}{n}\sum_{i=1}^{n} X_i.$$

Dermed

$$\widehat{\lambda}^2 \cdot \frac{1}{n}\sum_{i=1}^{n} X_i^2 = \widehat{\lambda} \cdot \frac{1}{n}\sum_{i=1}^{n} X_i \left(1 + \widehat{\lambda} \cdot \frac{1}{n}\sum_{i=1}^{n} X_i\right)$$

$$\Rightarrow \quad \widehat{\lambda} \cdot \left[\frac{1}{n}\sum_{i=1}^{n} X_i^2 - \left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)^2\right] = \frac{1}{n}\sum_{i=1}^{n} X_i$$

$$\Rightarrow \quad \underline{\underline{\widehat{\lambda} = \frac{\frac{1}{n}\sum_{i=1}^{n} X_i}{\frac{1}{n}\sum_{i=1}^{n} X_i^2 - (\frac{1}{n}\sum_{i=1}^{n} X_i)^2}}} \quad \text{og} \quad \underline{\underline{\widehat{r} = \frac{(\frac{1}{n}\sum_{i=1}^{n} X_i)^2}{\frac{1}{n}\sum_{i=1}^{n} X_i^2 - (\frac{1}{n}\sum_{i=1}^{n} X_i)^2}}}.$$

Oppgave 3

a)

Rimelighetsfunksjonen blir

$$L(\theta) = \prod_{i=1}^{n} f_X(x_i, \theta) = \prod_{i=1}^{n} \left[\frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{\theta}} e^{-\frac{1}{2\theta}(x_i-1)^2} \right];$$

log-rimelighetsfunskjonen blir

$$l(\theta) = \sum_{i=1}^{n} \left[-\frac{1}{2}\ln(2\pi) - \frac{1}{2}\ln\theta - \frac{1}{2\theta}(x_i-1)^2 \right] = -\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln(\theta) - \frac{1}{2\theta}\sum_{i=1}^{n}(x_i-1)^2.$$

Deriverer for å finne maksimumspunkt,

$$l'(\theta) = -\frac{n}{2\theta} + \frac{1}{2\theta^2}\sum_{i=1}^{n}(x_i-1)^2.$$

Finner SME ved å sette de partiellderiverte lik null,

$$l'(\theta) = 0 \quad \Rightarrow \quad -n = \frac{1}{\theta}\sum_{i=1}^{n}(x_i-1)^2 \quad \Rightarrow \quad \theta = \frac{1}{n}\sum_{i=1}^{n}(x_i-1)^2.$$

Dermed

$$\widehat{\theta} = \frac{1}{n}\sum_{i=1}^{n}(X_i-1)^2.$$

For at

$$\frac{n\widehat{\theta}}{\theta} = \sum_{i=1}^{n}\left(\frac{X_i-1}{\sqrt{\theta}}\right)^2,$$

vet man at

$$X_i \sim N(1,\theta) \quad \Rightarrow \quad \frac{X_i-1}{\sqrt{\theta}} \sim N(0,1)$$

$$\Rightarrow \quad \left(\frac{X_i-1}{\sqrt{\theta}}\right)^2 \sim \chi_1^2, \quad \text{og dermed} \quad \sum_{i=1}^{n}\left(\frac{X_i-1}{\sqrt{\theta}}\right)^2 \sim \chi_n^2,$$

fordi X_i-ene er uavhengige.

b)

Vet fra a) at $\frac{n\widehat{\theta}}{\theta} \sim \chi^2_n$.
Dette gir

$$\mathrm{E}\left(\frac{n\widehat{\theta}}{\theta}\right) = n \quad \Leftrightarrow \quad \frac{n}{\theta}\,\mathrm{E}\big[\,\widehat{\theta}\,\big] = n \quad \Rightarrow \quad \underline{\underline{\mathrm{E}\big[\,\widehat{\theta}\,\big] = \theta}},$$

$$\mathrm{Var}\left(\frac{n\widehat{\theta}}{\theta}\right) = n \quad \Leftrightarrow \quad \frac{n^2}{\theta^2}\,\mathrm{Var}\big[\,\widehat{\theta}\,\big] = 2n \quad \Rightarrow \quad \underline{\underline{\mathrm{Var}\big[\,\widehat{\theta}\,\big] = \frac{2\theta^2}{n}}}.$$

Benytter Cramer-Rao:

$$f_X(x;\theta) = \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{\theta}}\,e^{-\frac{1}{2\theta}(x-1)^2}$$

$$\ln f_X(x;\theta) = -\frac{1}{2}\ln(2\pi) - \frac{1}{2}\ln\theta - \frac{1}{2\theta}\,(x-1)^2$$

$$\frac{\partial}{\partial\theta}\ln f_X(x;\theta) = -\frac{1}{2\theta} + \frac{1}{2\theta^2}(x-1)^2$$

$$\frac{\partial^2}{\partial\theta^2}\ln f_X(x;\theta) = \frac{1}{2\theta^2} - \frac{2}{2\theta^3}(x-1)^2$$

$$\mathrm{E}\left[\frac{\partial^2}{\partial\theta^2}\ln f_X(X;\theta)\right] = \mathrm{E}\left[\frac{1}{2\theta^2} - \frac{2}{2\theta^3}(X-1)^2\right] = \frac{1}{2\theta^2} - \frac{1}{\theta^3}\mathrm{Var}[X] = \frac{1}{2\theta^2} - \frac{1}{\theta^3}\cdot\theta = -\frac{1}{2\theta^2}$$

$$\left\{-n\mathrm{E}\left[\frac{\partial^2}{\partial\theta^2}\ln f_X(X;\theta)\right]\right\}^{-1} = \frac{2\theta^2}{n} = \mathrm{Var}[\widehat{\theta}]$$

Enhver forventningsrett estimator $\tilde{\theta}$, for θ har $\mathrm{Var}[\tilde{\theta}] \geq \mathrm{Var}[\widehat{\theta}]$.
Derfor er $\widehat{\theta}$ en beste estimator.

c)

Vet at $V = \frac{n\widehat{\theta}}{\theta} \sim \chi^2_n$ slik at

$$\mathrm{P}\left(\chi^2_{1-\frac{\alpha}{2},n} \leq \frac{n\widehat{\theta}}{\theta} \leq \chi^2_{\frac{\alpha}{2},n}\right) = 1-\alpha$$

Men

$$\chi^2_{1-\frac{\alpha}{2},n} \leq \frac{n\widehat{\theta}}{\theta} \quad \Leftrightarrow \quad \theta \leq \frac{n\widehat{\theta}}{\chi^2_{1-\frac{\alpha}{2},n}}$$

$$\frac{n\widehat{\theta}}{\theta} \leq \chi^2_{\frac{\alpha}{2},n} \quad \Leftrightarrow \quad \frac{n\widehat{\theta}}{\chi^2_{\frac{\alpha}{2},n}} \leq \theta$$

dvs.

$$\mathrm{P}\left(\frac{n\widehat{\theta}}{\chi^2_{\frac{\alpha}{2},n}} \leq \theta \leq \frac{n\widehat{\theta}}{\chi^2_{1-\frac{\alpha}{2},n}}\right) = 1-\alpha,$$

og $(1-\alpha)100\%$ konfidensintervall for θ blir: $\underline{\underline{\left[\frac{n\widehat{\theta}}{\chi^2_{\frac{\alpha}{2},n}};\quad \frac{n\widehat{\theta}}{\chi^2_{1-\frac{\alpha}{2},n}}\right]}}$.

d)

Det er rimelig å forkaste H_0 dersom $\widehat{\theta} \leq k_l$ eller $\widehat{\theta} \geq k_r$.
Bestemmer k_l og k_r fra kravene:

$$\mathrm{P}\Big(\widehat{\theta} \leq k_l \big| H_0\Big) = \frac{\alpha}{2}, \quad \mathrm{P}\Big(\widehat{\theta} \geq k_r \big| H_0\Big) = \frac{\alpha}{2}$$

som er det samme med

$$\mathrm{P}\left(\frac{n\widehat{\theta}}{\theta} \leq \frac{nk_l}{1} | H_0\right) = \frac{\alpha}{2}, \quad \mathrm{P}\left(\frac{n\widehat{\theta}}{\theta} \geq \frac{nk_r}{1} | H_0\right) = \frac{\alpha}{2}$$

dvs.

$$nk_l = \chi^2_{1-\frac{\alpha}{2},n} \quad \text{og} \quad nk_r = \chi^2_{\frac{\alpha}{2},n}$$

$$\Rightarrow \quad k_l = \frac{1}{n}\,\chi^2_{1-\frac{\alpha}{2},n}, \quad k_r = \frac{1}{n}\,\chi^2_{\frac{\alpha}{2},n}$$

Derfor, er beslutningsregelen som

$$\text{“Forkast } H_0 \text{ dersom } \underline{\underline{\widehat{\theta} \leq \frac{1}{n}\,\chi^2_{1-\frac{\alpha}{2},n} \text{ eller } \widehat{\theta} \geq \frac{1}{n}\,\chi^2_{\frac{\alpha}{2},n}}}\text{”.}$$

Styrkefunksjonen blir

$$1-\beta(\theta) = \mathrm{P}\big(\text{Forkast } H_0|\theta\big) = \mathrm{P}\left(\widehat{\theta} \leq \frac{1}{n}\,\chi^2_{1-\frac{\alpha}{2},n}\,\middle|\,\theta\right) + \mathrm{P}\left(\widehat{\theta} \geq \frac{1}{n}\,\chi^2_{\frac{\alpha}{2},n}\,\middle|\,\theta\right)$$

$$= \mathrm{P}\left(\frac{n\widehat{\theta}}{\theta} \leq \frac{\chi^2_{1-\frac{\alpha}{2},n}}{\theta}\,\middle|\,\theta\right) + \mathrm{P}\left(\frac{n\widehat{\theta}}{\theta} \geq \frac{\chi^2_{\frac{\alpha}{2},n}}{\theta}\,\middle|\,\theta\right)$$

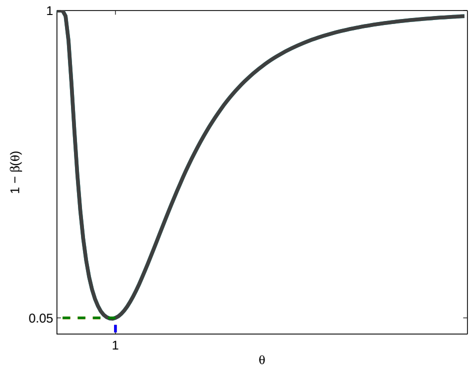
$$\underline{\underline{1-\beta(\theta) = \mathrm{P}\left(V \leq \frac{\chi^2_{1-\frac{\alpha}{2},n}}{\theta}\right) + \mathrm{P}\left(V \geq \frac{\chi^2_{\frac{\alpha}{2},n}}{\theta}\right)}}, \quad \text{når } V \sim \chi^2_n.$$

Med $\alpha = 0.05$ og $n = 10$ blir

$$\chi^2_{1-\frac{\alpha}{2},n} = 3.247, \quad \chi^2_{\frac{\alpha}{2},n} = 20.483$$

$$1 - \beta(\theta) = \mathrm{P}\Big(V \le \frac{3.247}{\theta}\Big) + \mathrm{P}\Big(V \ge \frac{20.483}{\theta}\Big).$$

Kvalitativ oppførsel



e)

Her

$$L(\theta) = \frac{1}{(2\pi)^{n/2}} \cdot \frac{1}{\theta^{n/2}} e^{-\frac{1}{2\theta}\sum_{i=1}^n (x_i-1)^2}$$

$\omega = \{\theta | \theta = 1\} = \{1\}$
$\Omega = \{\theta | \theta \neq 1\}$

$$L(\widehat{\omega}) = L(1) = \frac{1}{(2\pi)^{n/2}} \cdot \frac{1}{1^{n/2}} e^{-\frac{1}{2}\sum_{i=1}^n (x_i-1)^2} = \frac{1}{(2\pi)^{n/2}}\ e^{-\frac{1}{2}\sum_{i=1}^n (x_i-1)^2}$$

$$L(\widehat{\Omega}) = L(\widehat{\theta}) = \frac{1}{(2\pi)^{n/2}} \cdot \frac{1}{(\widehat{\theta})^{n/2}}\ e^{-\frac{1}{2\widehat{\theta}}\sum_{i=1}^n (x_i-1)^2} = \frac{1}{(2\pi\widehat{\theta})^{n/2}}\ e^{-\frac{n}{2}}, \quad \text{fordi } \widehat{\theta} = \frac{1}{n}\sum_{i=1}^n (x_i-1)^2$$

Dermed

$$\lambda = \frac{L(\widehat{\omega})}{L(\widehat{L(\Omega)})} = (\widehat{\theta})^{n/2} e^{-\frac{1}{2}\left[\sum_{i=1}^n (x_i-1)^2 - n\right]} = (\widehat{\theta})^{n/2} e^{-\frac{1}{2}\left[n\widehat{\theta} - n\right]} = \underline{\underline{(\widehat{\theta})^{n/2} e^{-\frac{1}{2}(\widehat{\theta}-1)}}}.$$

Testen i d) er **ikke** en GLRT fordi $\lambda(\theta) = \theta^{n/2} e^{-\frac{n}{2}(\theta-1)}$ er **ikke** en monoton funksjon av θ. Dette kan man for eksempel se ved å observere at

$$\frac{\partial}{\partial\theta}\left(\ln\lambda(\theta)\right) > 0 \quad \text{for} \quad \theta < 1$$

$$\frac{\partial}{\partial\theta}\left(\ln\lambda(\theta)\right) < 0 \quad \text{for} \quad \theta > 1.$$