Stochastic population modeling

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Fig 2.2

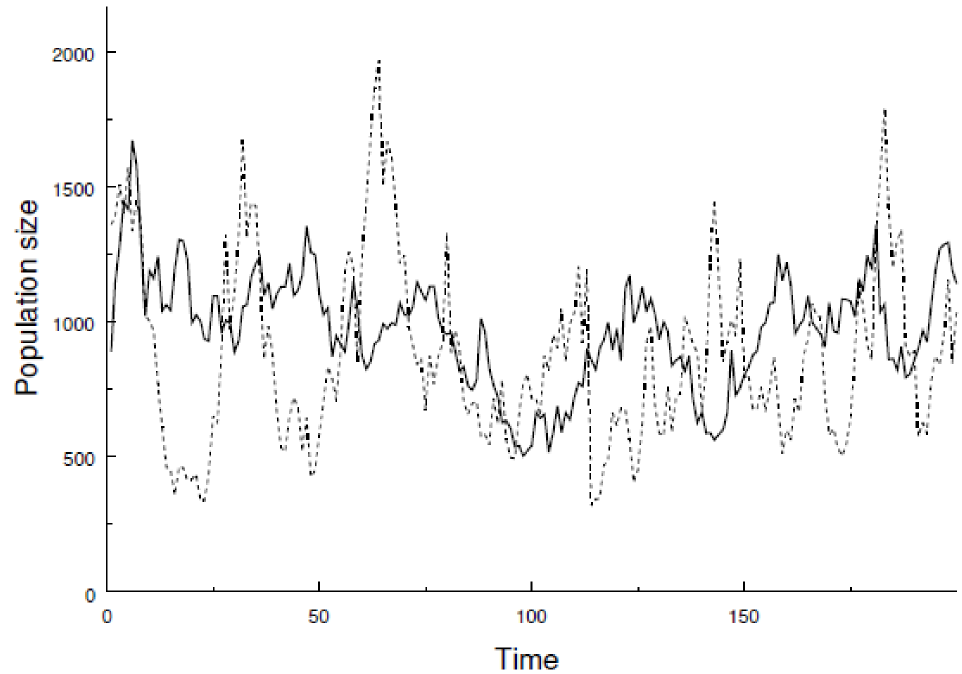
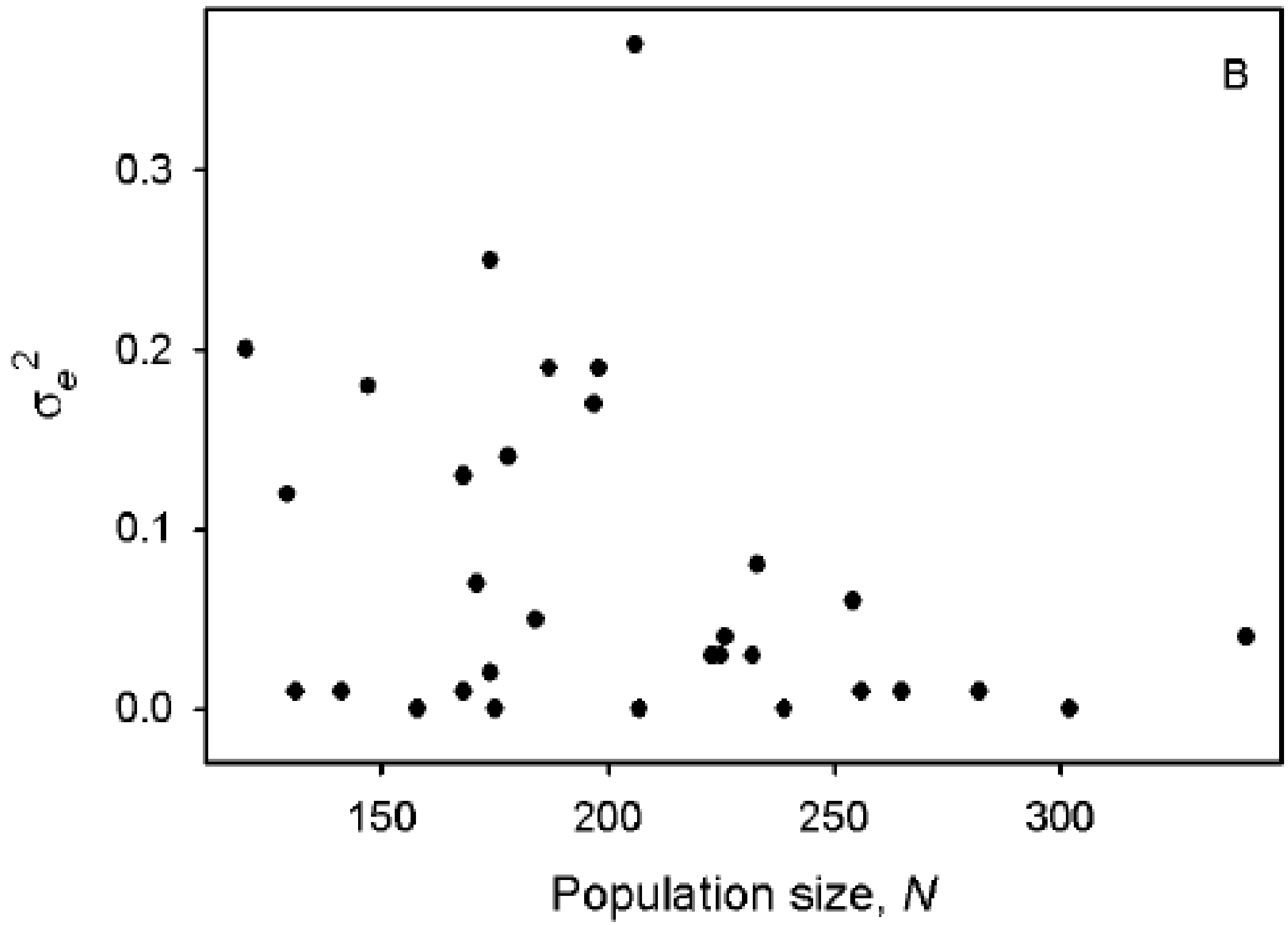


Figure 2.2: Simulations of the multiplicative process with $\sigma^2 = 0.01$ (solid line) and 0.04 (dotted line). The other parameters are $K = 1000$ and $\gamma = r = 0.1$, corresponding to a return time to equilibrium $T_R = 1/\gamma = 10$.



B
0.3
0.2
0.1
0.0
σ_e^2
150
200
250
300
Population size, N

3.2 Mean and variance for discrete processes

- No density dependence

$$X_t = X_0 + \sum_{i=0}^{t-1} S_i, \quad X_t \mid X_0 \sim N\left(X_0 + \mu t, \nu t\right)$$

- Density regulation

$$E\left(S_t \mid X_t = x\right) = \mu\left(x\right), \quad Var\left(S_t \mid X_t = x\right) = \nu\left(x\right)$$

Fig 3.1

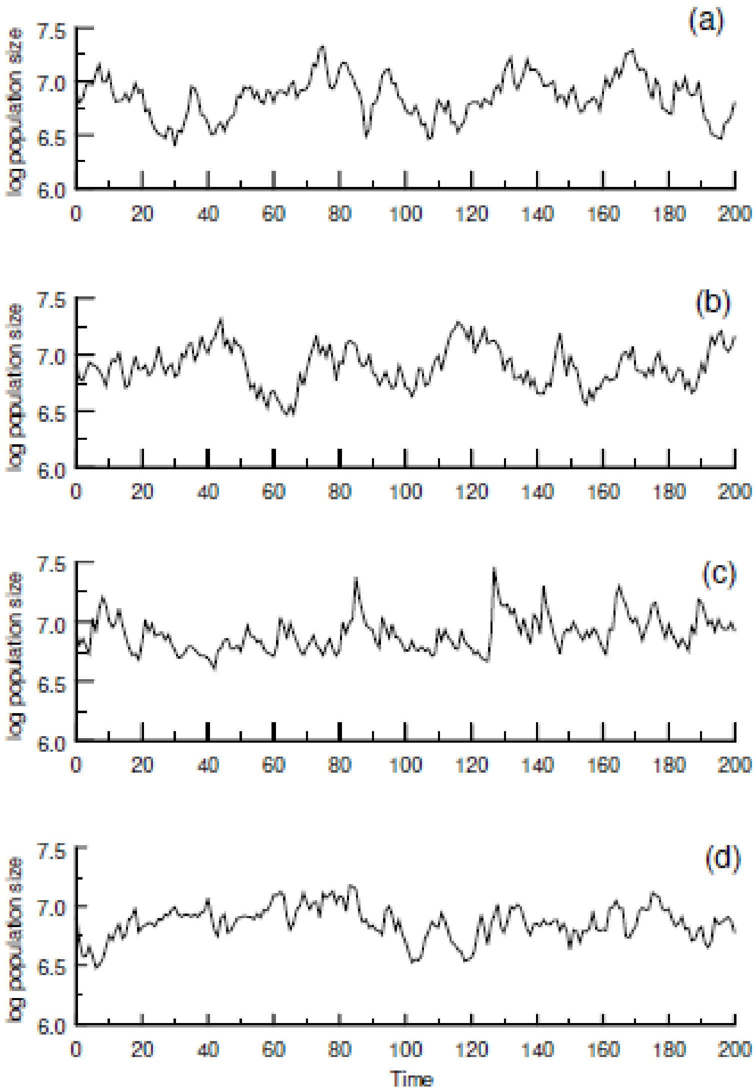


Figure 3.1: Population fluctuations for three models with the same discrete logistic type of dynamics with parameters $r = 0.2$, $K = 1000$, $\sigma_e^2 = 0.01$. The increments are modelled by different distributions: Normal distribution (a), Rectangular distribution (b), Exponential distribution (c) and the diffusion approximation recorded at discrete values with increments 1 (d).

3.3 Diffusion – infinitesimal mean and variance

- For constant $\mu(x)=\mu$ and $\nu(x)=\nu$ we have for all t:

$$E\left(X_t - X_0 \mid X_0\right) = \mu t$$
$$Var\left(X_t - X_0 \mid X_0\right) = \nu t$$

- Diffusion = this hold for $t \rightarrow 0$

$$E\left(\Delta X_t \mid X_t = x\right) = \mu(x)\Delta t$$
$$Var\left(\Delta X_t \mid X_t = x\right) = \nu(x)\Delta t$$

Diffusion process - definition

- Inifinitesimal mean:

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t} E\left(\Delta X_t \mid X_t = x\right) = \mu\left(x\right)$$

- Infinitesimal variance:

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t} E\left(\left(\Delta X_t\right)^2 \mid X_t = x\right) = \nu\left(x\right)$$

- and some boundary conditions (e.g. absorbing barrier at extinction).

Diffusion process

Ito approach:

Choose the expectation and variance functions of the discrete process as infinitesimal mean and vaiance.

Diffusion approximation for N_t?

$$\mu_N(n) = E(\Delta N \mid N = n)$$

$$\nu_N(n) = Var(\Delta N \mid N = n) \approx E(\Delta N^2 \mid N = n)$$

These tow diffusions not quite equal!

Fig 3.2

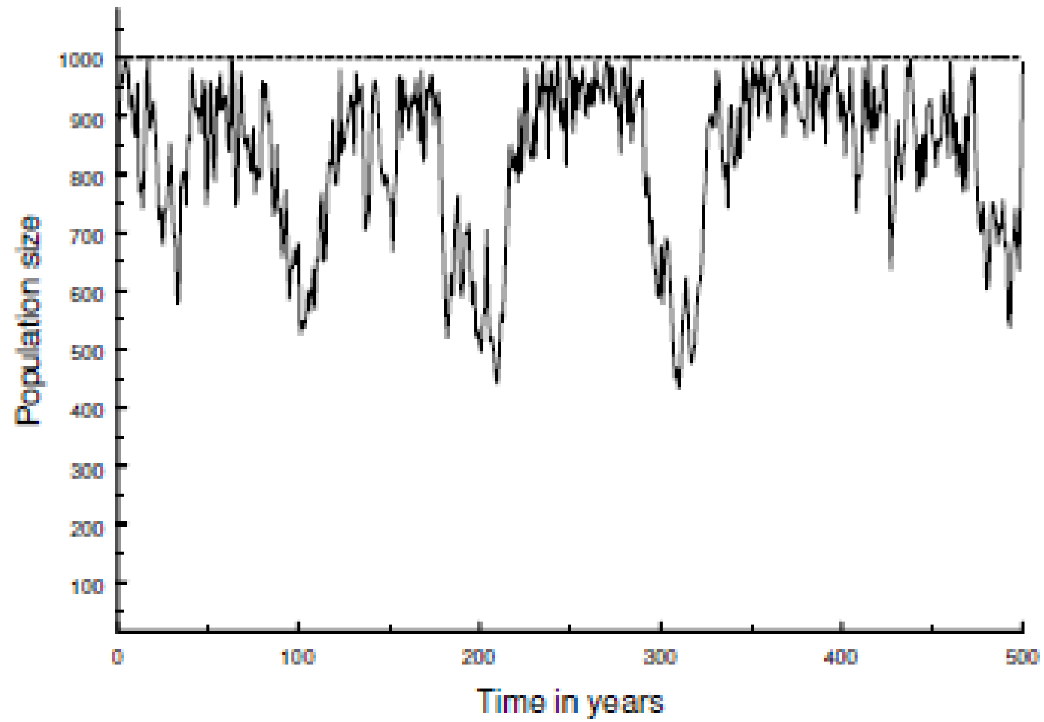


Figure 3.2: Simulation of the ceiling model with reflecting barrier at population size 1000. The parameters are $r_1 = 0.02$, $\sigma_e^2 = 0.01$.

3.5 Transformations

If N_t is a diffusion process, then also $X_t = g(N_t)$ is a diffusion for any function $g(\bullet)$ that is twice differentiable.

Infinitesimal mean for $X_t = g(N_t)$:

$$\mu_X(x) = g'(n)\mu_N(n) + \frac{1}{2}g''(n)\nu_N(n), \quad n = g^{-1}(x)$$

Infinitesimal variance:

$$\nu_X(x) = Var(\Delta X_t \mid X_t = x) = g'(n)^2 \nu_N(n)$$

Brownian motion and OU-process

- Brownian motion: Diffusion process with constant infinitesimal mean and variance.
- Ornstein-Uhlenbeck:
  - Infinitesimal mean linear expression
  - Infinitesimal variance constant (ignore σ_d^2)

Fig 3.3

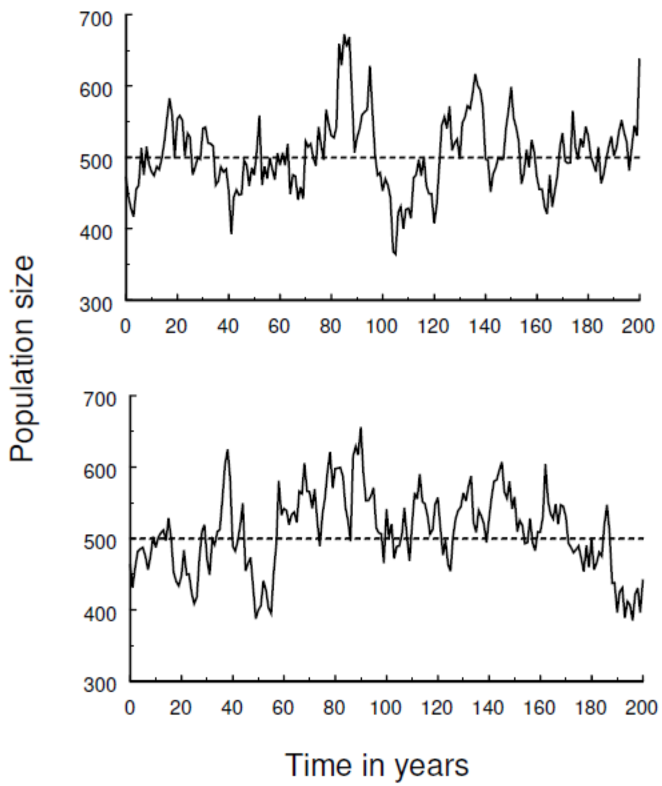


Figure 3.3: Simulation of the theta-logistic model with $r_1 = 0.1$, $K = 500$, $\sigma_e^2 = 0.005$ and $\theta = 2$ (upper panel) and the same process approximated by an Ornstein-Uhlenbeck process (lower panel).

3.8.1 The Green function $G(x,x_0)$ for a diffusion process

The expected time the process is in $(x,x+\Delta x)$ before reaching an absorbing barrier is $G(x,x_0)\Delta x$ when $\Delta x \to 0$.

Expected time in [c,d]: $\int_c^d G(x,x_0)\,dx$

Mathematical expressions for $G(x,x_0)$

$$s(x) = \exp\left[-2\int^{x} \frac{\mu(z)}{\nu(z)}dz\right] \quad \text{and} \quad S(x) = \int^{x} s(z)\,dz$$

Help function (to simplify notation): $m(x) = \dfrac{1}{\nu(x)s(x)}$,

$$G(x,x_0) = \begin{cases} 2\dfrac{\left[S(x)-S(a)\right]\left[S(b)-S(x_0)\right]}{S(b)-S(a)}m(x) & a \le x \le x_0 \le b \\ 2\dfrac{\left[S(b)-S(x)\right]\left[S(x_0)-S(a)\right]}{S(b)-S(a)}m(x) & a \le x_0 \le x \le b \end{cases}$$