Stochastic population modeling

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3.8.1 The Green function $G(x,x_0)$ for a diffusion process

The expected time the process is in $(x,x+\Delta x)$ before reaching an absorbing barrier is $G(x,x_0)\Delta x$ when $\Delta x \to 0$.

Expected time in [c,d]: $$\int_c^d G(x,x_0)\,dx$$

Mathematical expressions for $G(x,x_0)$

$$s(x) = \exp\left[-2\int^{x} \frac{\mu(z)}{\nu(z)}dz\right]$$ and $$S(x) = \int^{x} s(z)dz$$

Help function (to simplify notation): $m(x) = \dfrac{1}{\nu(x)s(x)}$,

$$G(x,x_0) = \begin{cases} 2\dfrac{[S(x)-S(a)][S(b)-S(x_0)]}{S(b)-S(a)}m(x) & a \le x \le x_0 \le b \\ 2\dfrac{[S(b)-S(x)][S(x_0)-S(a)]}{S(b)-S(a)}m(x) & a \le x_0 \le x \le b \end{cases}$$

Exponential growth with demographic variance

$$\mu_N(n) = rn$$
$$\nu_N(n) = \sigma_d^2 n + \sigma_e^2 n^2$$

$$\Rightarrow s(n) = \left(\frac{\sigma_d^2 + \sigma_e^2}{\sigma_d^2 + \sigma_e^2 n}\right)^{\frac{2r}{\sigma_e^2}}$$

$$\Rightarrow S(n) = \begin{cases} \dfrac{\sigma_d^2 + \sigma_e^2}{2s}\left[1 - \left(\dfrac{\sigma_d^2 + \sigma_e^2 n}{\sigma_d^2 + \sigma_e^2}\right)^{-\frac{2s}{\sigma_e^2}}\right] & s \neq 0 \\ \dfrac{\sigma_d^2 + \sigma_e^2}{2s} \ln\left(\dfrac{\sigma_d^2 + \sigma_e^2 n}{\sigma_d^2 + \sigma_e^2}\right) & s = 0 \end{cases}$$

With $b = \infty$, P(extinction)=1 for s<0
and

$$P(ext) = \left(\frac{\sigma_d^2 + \sigma_e^2}{\sigma_d^2 + \sigma_e^2 n_0}\right)^{\frac{2s}{\sigma_e^2}} \qquad s > 0$$

Fig. 3.4

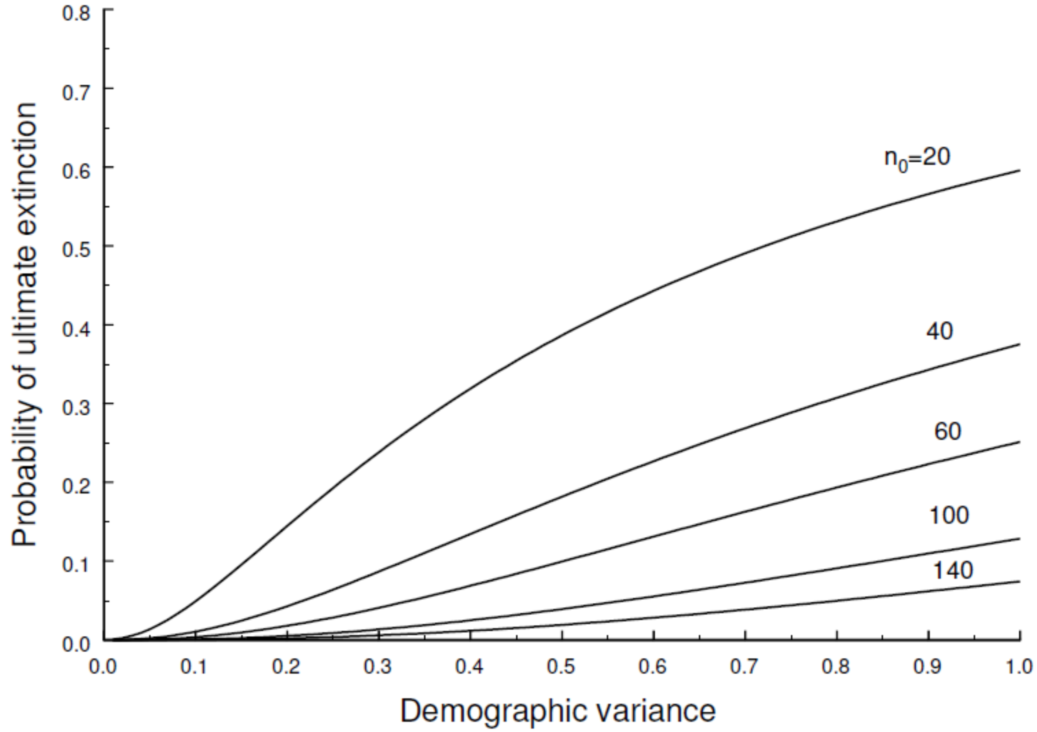

Figure 3.4: The probability of ultimate extinction as function of the demographic variance for different initial population sizes. The diffusion has infinitesimal mean $\mu(n) = rn$ with $r = 0.015$ and constant environmental and demographic variance, $\sigma_e^2 = 0.01$.

Expected time to extinction, T

For populations with density regulation, let a=1 and b infinity:

$$E(T) = \int_1^{n_0} 2m(n)S(n)\,dn + \int_{n_0}^{\infty} 2m(n)S(n_0)\,dn$$

Stationary distributions - OU

Adds density regulation (no extinction barrier):

$$\mu(x) = \alpha - \beta x$$
$$\nu(x) = \sigma^2$$

$$\Rightarrow X_t \mid X_0 = x_0 \sim N\left(\frac{\alpha}{\beta} + \left(x_0 - \frac{\alpha}{\beta}\right)e^{-\beta t}, \frac{\sigma^2}{2\beta}\left(1 - e^{-2\beta t}\right)\right)$$

Stationary distribution ($t \to \infty$): $f_\infty(x, x_0) = f(x)$

Here: $N\left(\frac{\alpha}{\beta}, \frac{\sigma^2}{2\beta}\right)$

Stationary distributions – theta-logistic model

$$\theta \neq 0, \quad \mu(n) = rn\left(1-\left(\frac{n}{K}\right)^{\theta}\right), \quad r = \frac{r_1}{\left(1-K^{-\theta}\right)}$$

$$\nu(n) = \sigma_e^2 n^2$$

$$\Rightarrow f(n;\alpha,K,\theta) = Cn^{\alpha-1}e^{-\frac{(\alpha+1)}{\theta}\left(\frac{n}{K}\right)^{\theta}}, \quad \alpha = \frac{2r}{\sigma_e^2} - 1 = \frac{2s}{\sigma_e^2} > 0$$

Transformation of variable: $y = \left(\frac{n}{K}\right)^{\theta}$

$$\Rightarrow f(n;\alpha,K,\theta) = \frac{|\theta|\left(\frac{\alpha+1}{\theta}\right)^{\alpha/\theta}}{K\Gamma(\alpha/\theta)}\left(\frac{n}{K}\right)^{\alpha-1} e^{-\frac{(\alpha+1)}{\theta}\left(\frac{n}{K}\right)^{\theta}}, \quad \theta \neq 0$$

This is called the generalized gamma dist.

Fig. 3.5

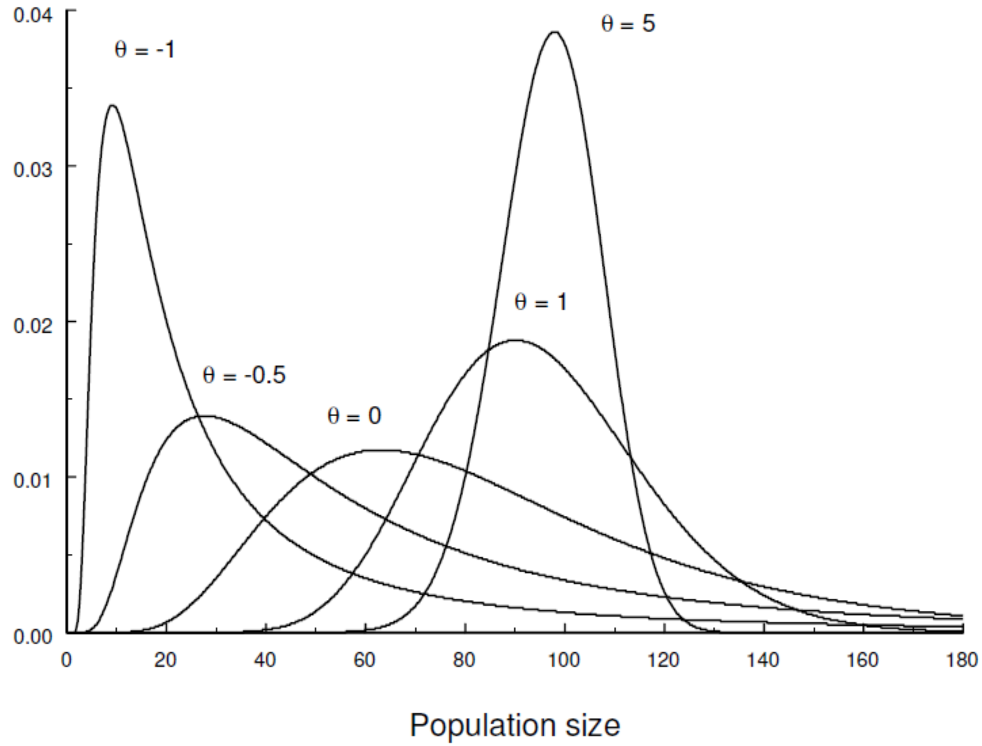

Figure 3.5: The stationary distribution for the theta-logistic model, the generalized gamma distribution, for different values of θ. The other parameters are $K = 100$, $\sigma_e^2 = 0.01$, and $r_1 = 0.1$.