Stochastic population modeling

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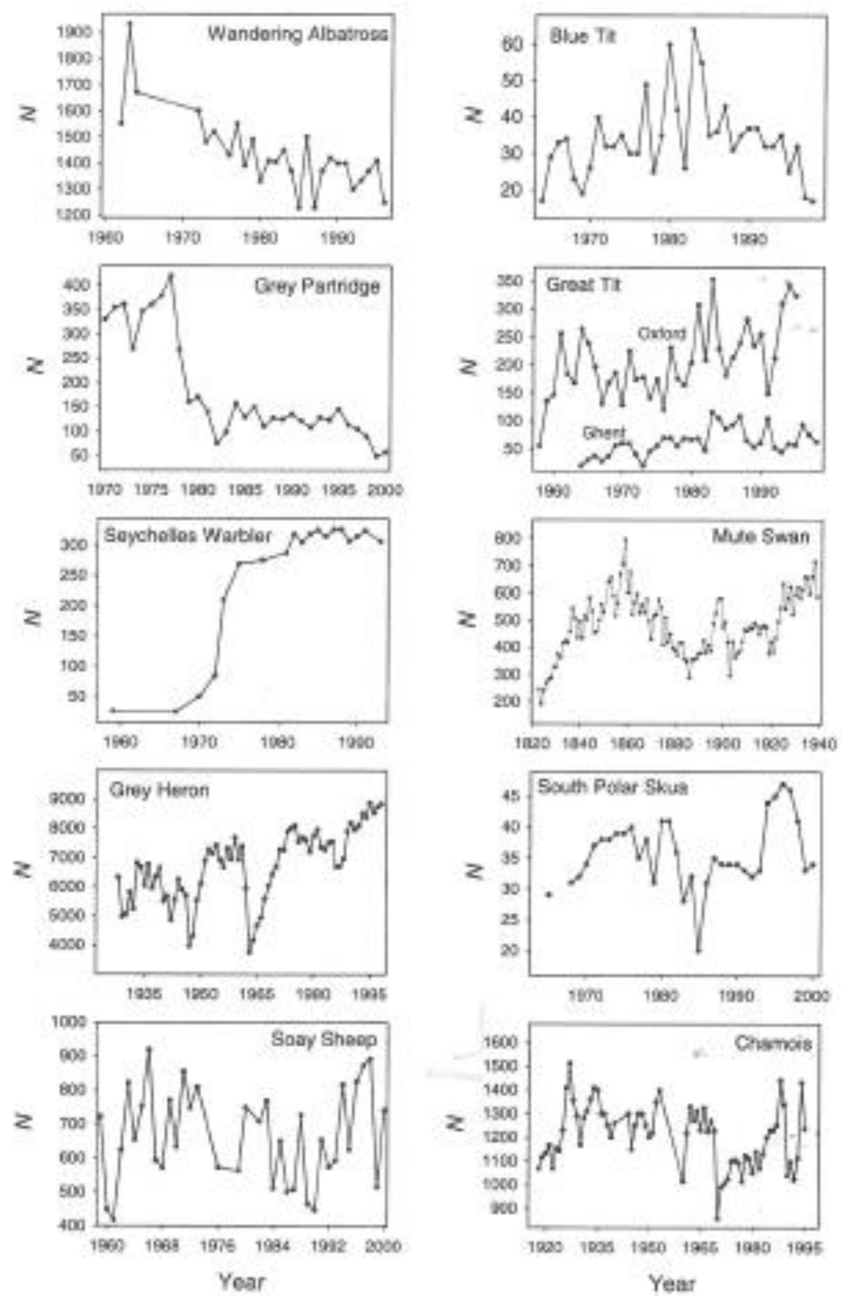
Some time-series



1.3 Lognormal distribution

Multiplicative effects common in biological processes -> lognormal distribution important

Central limit theorem:

$\sum_{u=0}^{t-1} S_u$ approx. normally disributed

-> $X_t = X_0 + \sum_{u=0}^{t-1} S_u \mid X_0$ also approx. normal

-> $N_t = N_0 e^{\sum S_u} \mid N_0$ approx. lognormal

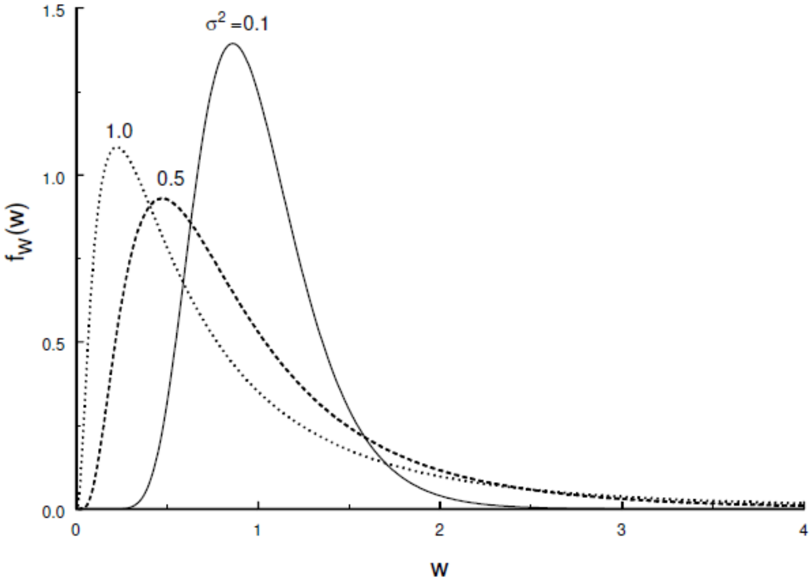
Lognormal distribution

$$Y \sim N\left(\mu, \sigma^2\right)$$

$$Y = \ln V \quad V = e^Y$$

$$V \sim LogN\left(\mu, \sigma^2\right)$$

$$f_V\left(v\right) = \frac{1}{\sqrt{2\pi}\sigma v} e^{-\frac{1}{2}\left(\frac{\ln v - \mu}{\sigma}\right)^2}, v > 0$$



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Stochastic growth rate

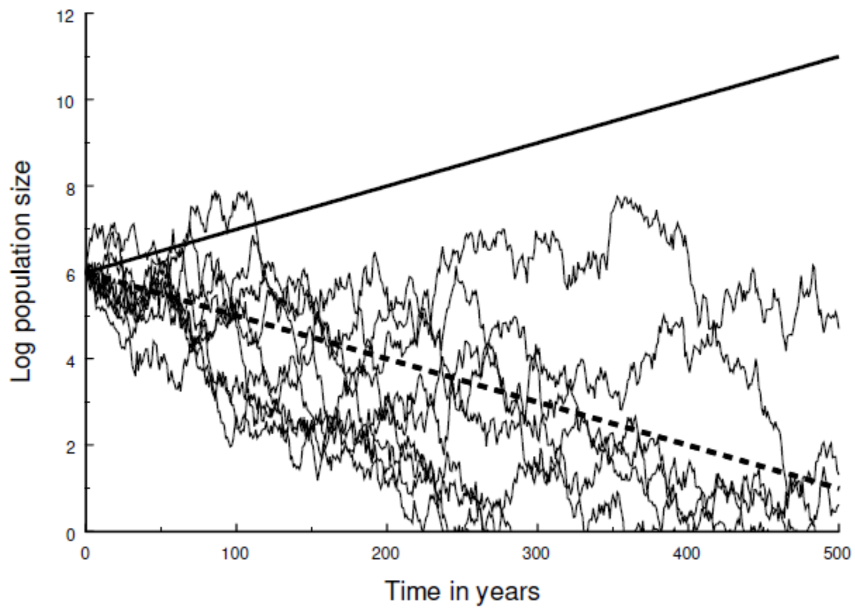

Figure 1.2: Simulation of 10 sample paths using the above stochastic model with $r = 0.01$ and $\sigma_e^2 = 0.04$. The solid straight line shows the deterministic growth, while the dotted line is the mean stochastic growth.

MLE of s?

$$\hat{s} = \bar{D} = \frac{1}{n}\sum_{i=1}^{n} D_i = \frac{X_n - X_0}{n}$$

$$\hat{\sigma}_s^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(D_i - \hat{s}\right)^2$$

Distribution?

$$\frac{\hat{\sigma}_s^2 (n-1)}{\sigma_s^2} \sim ? \qquad \chi^2_{df=n-1}$$

Prediction

$$\frac{X_{n+m} - X_n - m\hat{s}}{\sigma_s \sqrt{m + m^2 / n}} \sim N(0,1)$$

$$\frac{\hat{\sigma}_s^2 (n-1)}{\sigma_s^2} \sim \chi_{n-1}^2$$

$$\Rightarrow T = \frac{X_{n+m} - X_n - m\hat{s}}{\hat{\sigma}_s \sqrt{m + m^2 / n}} \sim T_{df=n-1}$$

Prediction interval

$$P\left(-t_{n-1,\alpha/2} < T_{n-1} < t_{n-1,\alpha/2}\right) = 1-\alpha$$

$(1-\alpha)$ prediction interval for X_{n+m}

$$X_n + m\hat{s} \pm t_{n-1,\alpha/2}\hat{\sigma}_s\sqrt{m + m^2/n}$$

Figure 1.3

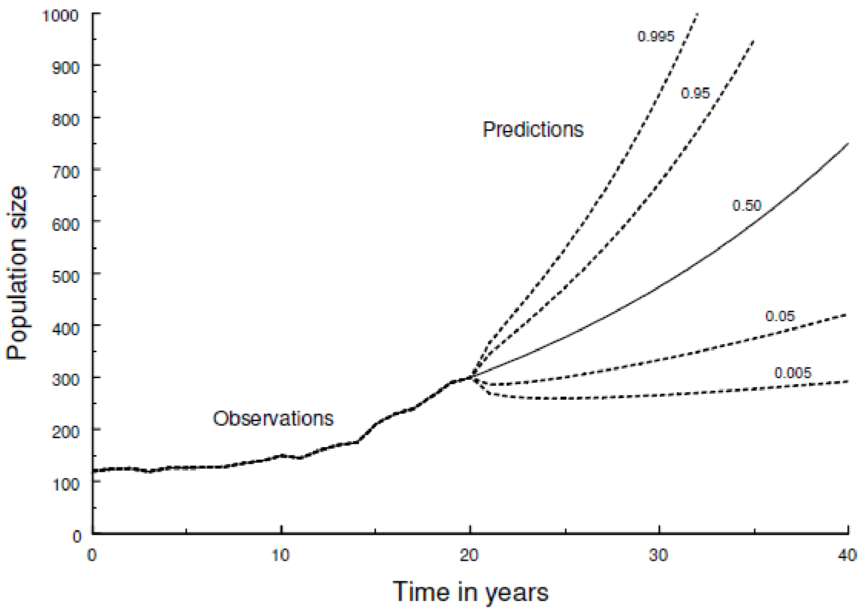

Figure 1.3: Time series observations of a population of Sea Eagles over 21 years together with 20 prediction intervals (90 and 99 %) and predicted median 20 years ahead.

1.6 Demographic stochasticity

- Assume environment (***Z***) constant – just look at differencens between individuals, and
- populations with only adult individuals with the same stochastic properties
- *N* individuals (females) then contribute with $w_1, w_2, \ldots, w_N$ to next year, where
  - w = #offspring $+I$, $I = \begin{cases} 1 & survive \\ 0 & die \end{cases}$
  - w = individual fitness
- Demographic variance $Var(w_i) = Var(d_i) = \sigma_d^2$

Conditionality

1. $E_Y\left[E_x\left(X \mid Y\right)\right] = E_X\left(X\right)$
2. $Var\left(X\right) = E_Y Var_X\left(X \mid Y\right) + Var_Y E_X\left(X \mid Y\right)$
3. $Cov\left(X,Y\right) = ECov\left(X,Y \mid Z\right) + Cov\left(E\left(X \mid Z\right), E\left(Y \mid Z\right)\right)$

Birth and death pro

- $W=I+B$, I and B indep

$$\sigma_d^2 = p(1-p) + \sigma_b^2$$

$$\lambda = p + b$$

$$r = \ln(p+b)$$

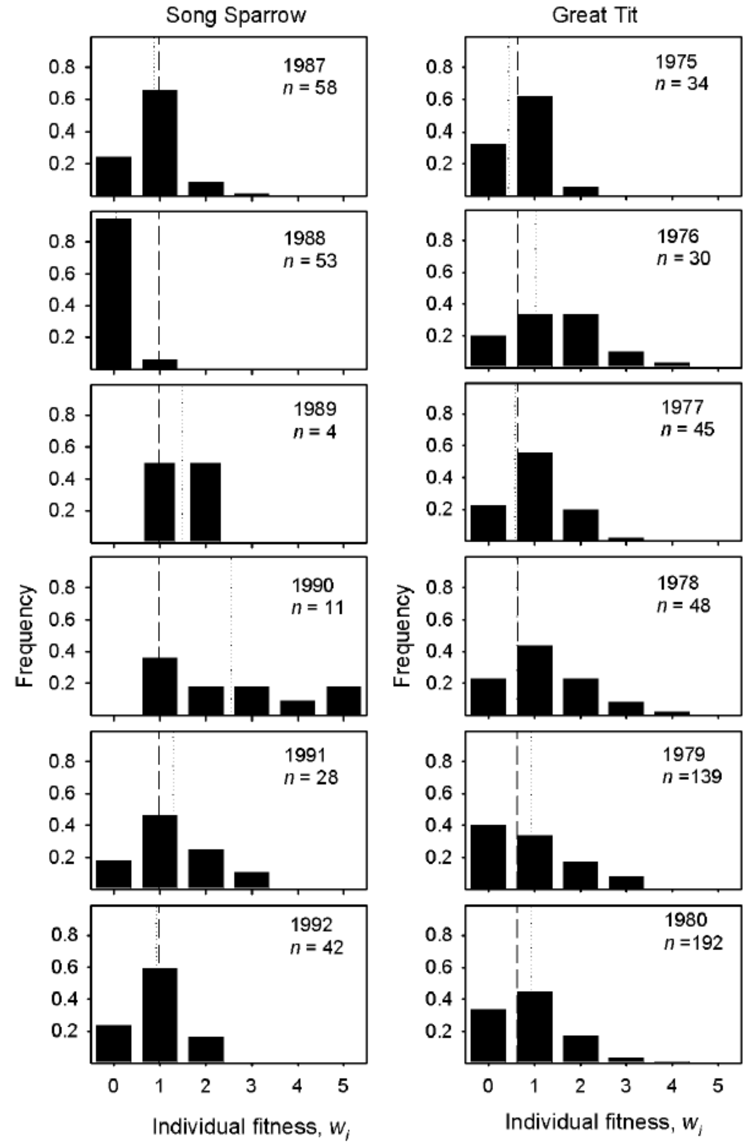

Figure 1.4: Annual variation in the distribution of contributions w to the next generation for two passerine species, the Song Sparrow Melospiza melodia on Mandarte island and the Great Tit in Wytham Wood. The dashed line indicates the mean values across all years and the dotted line the mean contribution a single year.

Quick repetition

- ***Z*** constant -> only demographic stochasticity

$$Var\left(\Delta N \mid N\right) = \sigma_d^2 N \quad Var\left(\Lambda\right) = \frac{\sigma_d^2}{N}$$

- Multiplicative model, large pop., ***Z*** stochastic

$$Var\left(\Delta N \mid N\right) = \sigma_e^2 N^2 \quad Var\left(\Lambda\right) = \sigma_e^2$$

1.7 Demographic and environmental stochasticity together

If ***Z*** fluctuates between years -> individual contributions no longer independent

-> $$w_i = Ew + e + d_i$$

where

$$e = E(w \mid z) - E(w)$$

and

$$d_i = w_i - E(w \mid z)$$

Fig 1.5

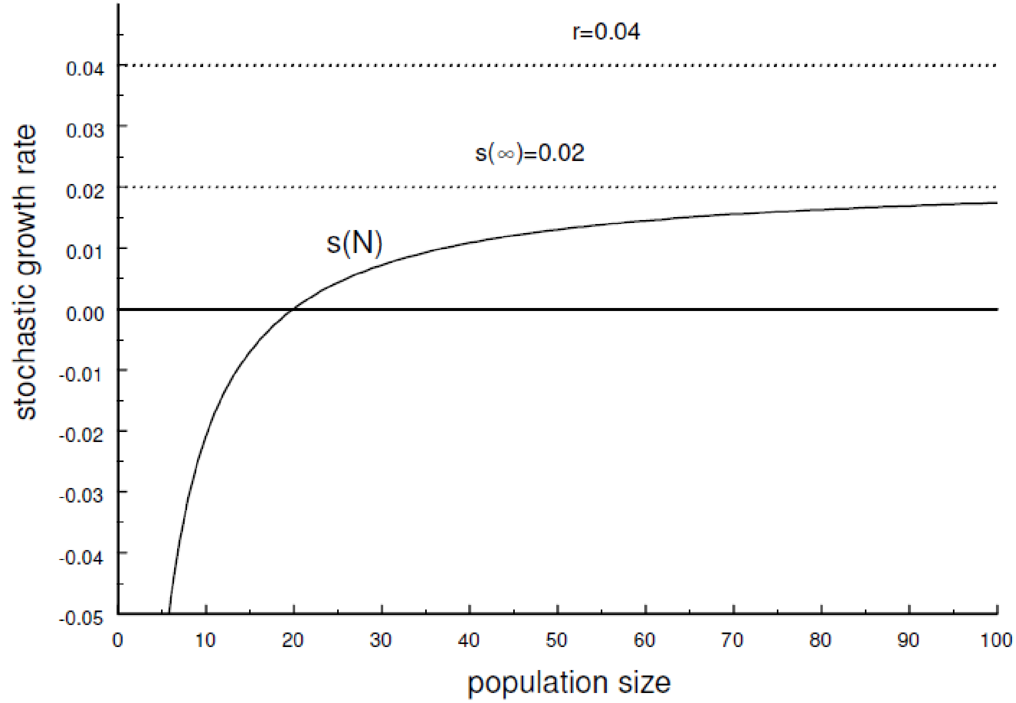

Figure 1.5: The stochastic growth rate as function of the population size. Parameter values are $r = \ln \mathrm{E}w = \ln \lambda = 0.04$, $\sigma^2 = 0.04$, and $\sigma_d^2 = 1$, giving $N^* \approx 20$. The dotted lines show the value of r and $s(\infty)$.

Fig 1.6

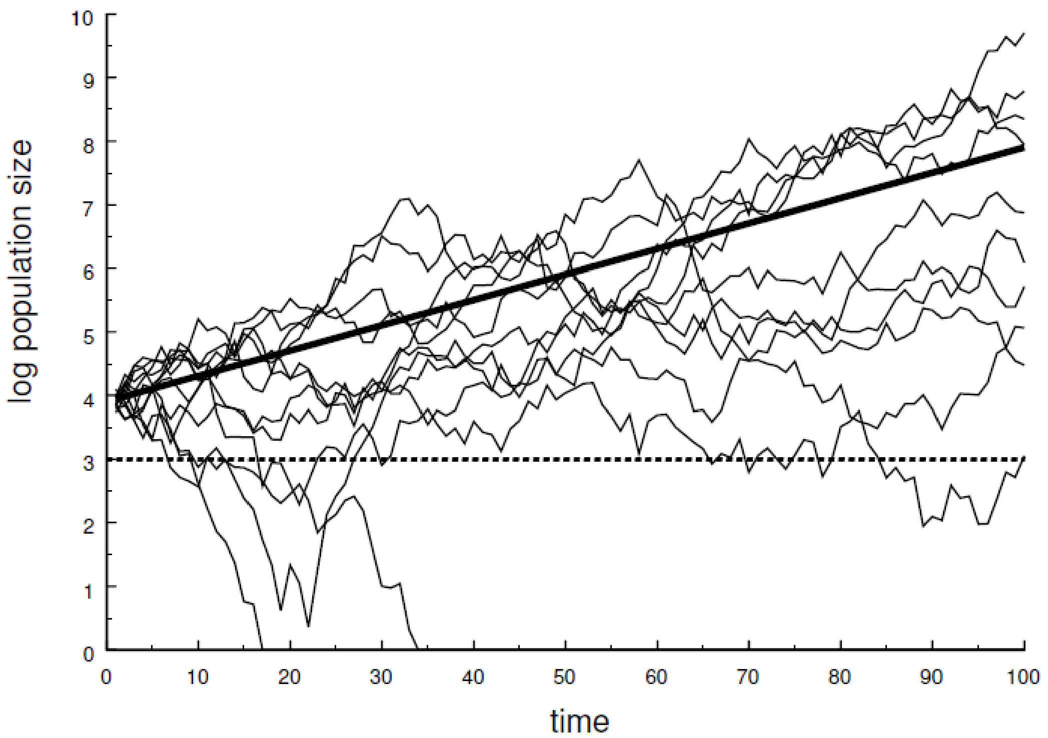

Figure 1.6: Simulations of the process described in Fig.1.5 with initial population size $N_0 = 50$ giving $\ln N_0 \approx 4$.

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