Stochastic population modeling

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31.03.2016

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4.2.3 Matrix formulation

- **Multiplicative growth rate** of the population is the dominant eigenvalue λ of the projection matrix $\underline{l}$
- **Stable age distribution** $\underline{u}$, $\left(\sum u_i = 1\right)$
  = right dominant eigenvector defined by $\underline{l\,u} = \lambda \underline{u}$
- **Reproductive values** $\underline{v}$, $\left(\sum u_i v_i = 1\right)$
  defined by $\underline{v\,l} = \lambda \underline{v}$
- **Total reproductive value** after one generation $V + \Delta V = \underline{v}\left(\underline{n} + \Delta \underline{n}\right) = \underline{v}\,\underline{l}\,\underline{n} = \lambda \underline{v n} = \lambda V$

4.2.3 Matrix formulation

Important result from linear algebra: $\dfrac{\partial \lambda}{\partial l_{i,j}} = v_i u_j$

where $l_{i,j}$ is a non-zero element in $\underline{l}$

This is called the sensitivity of λ with respect to the (i,j)th element

4.3.3 Reproductive value dynamics

$\underline{n}$: population vector

$\underline{v}$: reproductive value defined by

$$\underline{l} = E\underline{M} = EE\left(\underline{M} \mid \underline{Z}\right); \quad \underline{v\,l} = \lambda\underline{v}$$

$\underline{u}$: age structure, $\sum u_i = 1; \quad \underline{l\,u} = \lambda\underline{u}$

Scaling: $\sum u_i v_i = 1$

$V = \sum n_i v_i$: total reproductive value

Environmental and demographic variances

We then have:

$$\sigma^2 = \lambda^{-2}\sigma_V^2 = \sigma_e^2 + \sigma_d^2 / N$$

where

$$n_j \approx N \times u_j$$

$$\sigma_d^2 = \sum_{j=1}^{k} \lambda^{-2} u_j \left[v_1^2 \sigma_{B_j}^2 + v_{j+1}^2 \sigma_{P_j}^2 + 2 v_1 v_{j+1} \sigma_{BP_j}^2 \right]$$

Since $N \approx V$, the process for $\ln V_{t+1}$ has the same dynamics as a population without age structure.

Environmental and demographic variances

This process can then be approximated by a diffusion process with

inf. mean $$E(\Delta \ln V \mid V) = r - \frac{1}{2}\sigma_e^2 - \frac{1}{2V}\sigma_d^2$$

inf. var $$Var(\Delta \ln V \mid V) = \sigma_e^2 + \sigma_d^2 / V$$

Fig. 4.2

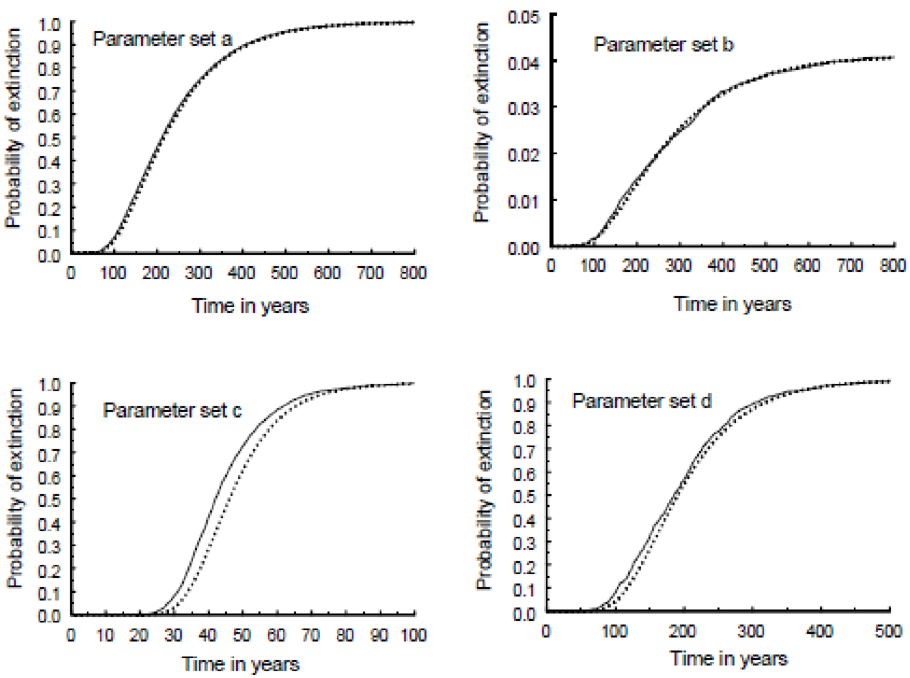

Figure 4.2: Cumulative distributions of time to extinction for processes without environmental stochasticity. Dotted lines are the cumulative distribution for the diffusion approximations, while the solid lines are based on stochastic simulations of the age-structured process. The individual yearly fecundity is restricted to take values 0 or 1. Demographic stochasticity is then uniquely determined by the elements of the projection matrix. The initial population size is 100. Parameter sets a and b are for populations with 4 age classes. For set a the survivals are (0.4, 0.6, 0.9, 0.7), the fecundities (0, 0.6, 0.9, 0.7). For set b the growth rate is slightly positive so that the probability of ultimate extinction is smaller than one. The survival rates in set b are the same as in set a while the fecundities are (0, 0.6, 0.9, 0.8). Parameter sets c and d are populations with 9 age-classes. The survivals in set c are (0.6, 0.7, 0.7, 0.8, 0.9, 0.9, 0.9, 0.7, 0.5), and the fecundities (0, 0, 0, 0.4, 0.5, 0.8, 0.8, 0.9, 0.8). The survivals in set d are (0.5, 0.7, 0.7, 0.8, 0.8, 0.9, 0.9, 0.7, 0.5) and the fecundities are (0, 0, 0, 0, 0.2, 0.5, 0.6, 0.9, 0.8).

Fig 4.3

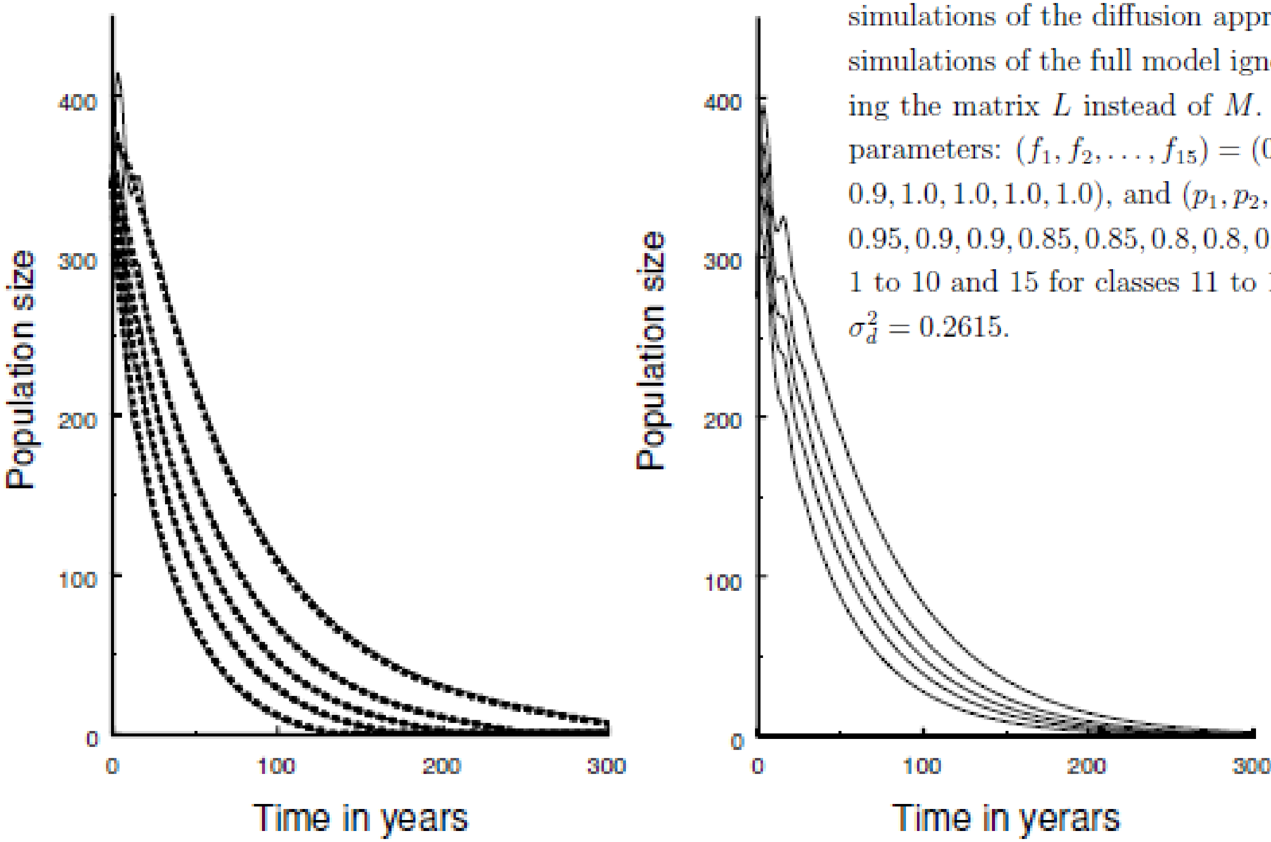

Figure 4.3: Stochastic simulations of an age-structured model with Poisson lognormally distributed number of offspring (exercise 7-9) with variance 3 times the mean value. The only parameter with environmental stochasticity is the first year survival P_1 assumed to have variance 0.04 between years. The solid lines from bottom to top in the left panel are the 0.05, 0.25, 0.50, 0.75 and 0.95 quantiles computed from 100,000 simulations of the full age-structured model while the dotted lines are from the corresponding simulations of the diffusion approximation. The right panel shows the same simulations of the full model ignoring demographic stochasticity, that is, using the matrix L instead of M. There are 15 age classes and the following parameters: $(f_1, f_2, \ldots, f_{15}) = (0, 0, 0, 0, 0, 0, 0, 0.5, 0.6, 0.7, 0.9, 1.0, 1.0, 1.0, 1.0)$, and $(p_1, p_2, \ldots, p_{15}) = (0.5, 0.6, 0.7, 0.9, 0.95, 0.95, 0.95, 0.95, 0.9, 0.9, 0.85, 0.85, 0.8, 0.8, 0.6)$, and initial population size 20 for classes 1 to 10 and 15 for classes 11 to 15. This gives $\lambda = 0.9813$, $\sigma_e^2 = 0.0011$, and $\sigma_d^2 = 0.2615$.