# Week 4: The Normal Distribution

# Recap

So far we have

- learned about maximising the likelihood
- estimated confidence intervals and standard errors

These are the basic tools we will use to fit and understand our models

# This Week

- More than one datum: the probability
- Some data: Punxsutawney Phil & Groundhog Day
- The Normal Distribution
  - the log likelihood
- Different Amounts of Data
- MLEs
  - The MLE for $\mu$
  - The MLE for $\sigma^2$
  - The distribution of $\hat{\mu}$
- Modeling: Predicting the End of Winter, with a t-test

# More than one datum

So far we have only used one data point. But what if we have more?

If we make one assumption, the maths is easy

# More than one datum: the probability

If data are independent, then

$$Pr(X_1, X_2) = Pr(X_1)Pr(X_2)$$

So we can multiply the probabilities

In general, then

$$Pr(X_1, X_2, ..., X_n) = \prod_{i=1}^{n} Pr(X_i)$$

# More than one datum: the likelihood

The log-likelihood for the parameters ($\theta$) as a function of the data is

$$l(\theta; X_1, X_2, ..., X_n) = \sum_{i=1}^{n} \log(Pr(X_i; \theta))$$

So we just add the log-likelihoods together

- easier than multiplying!

# Punxsutawney Phil & Groundhog Day

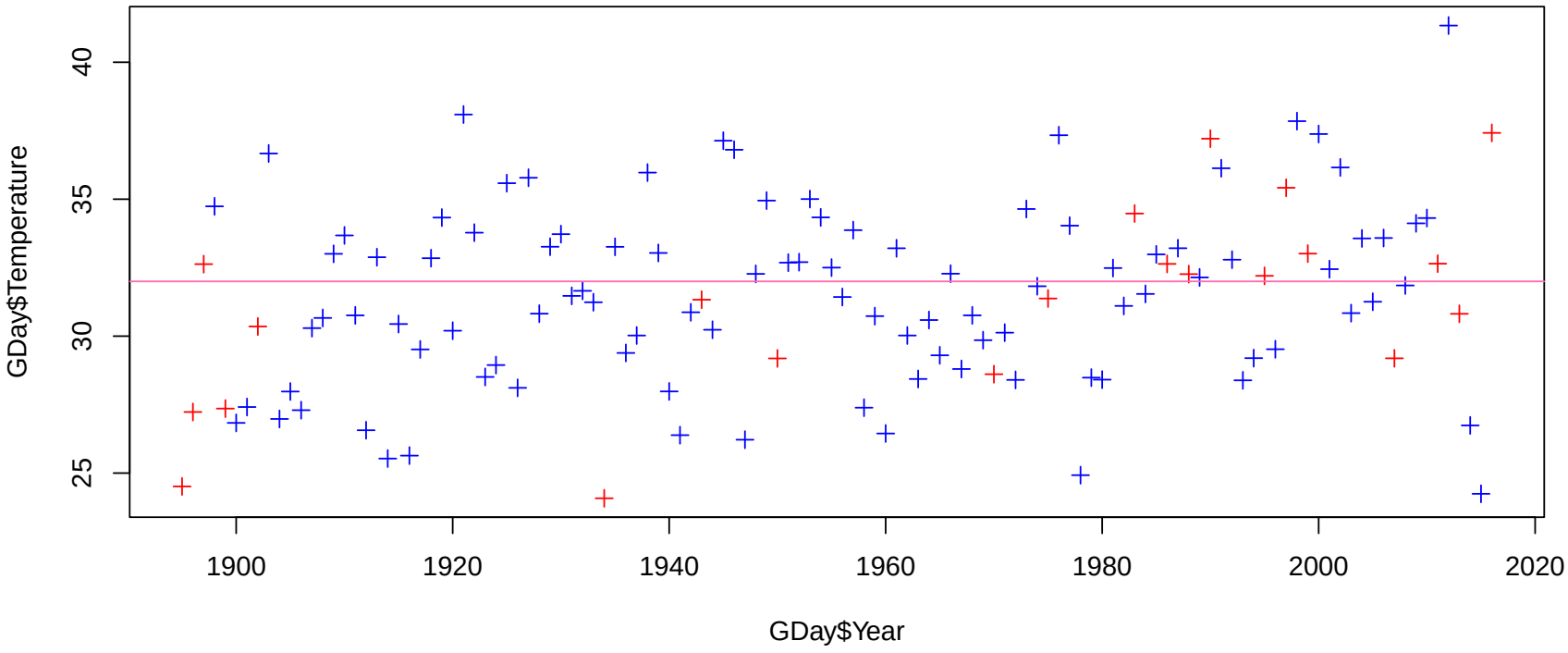
Here is some data on whether Punxsutawney Phil predicts another 6 weeks of winter (which he tries to do every Feb 2)

We will look at the average temperature for Feb/March in Pennsylvania in each year (web link: "https://www.math.ntnu.no/emner/ST2304/2024v/Module04/GroundhogData.csv")

```
GDay <- read.csv(file="GroundhogData.csv")
```

# Winter/Spring Temperatures

These are the mean February/March temperatures




Initially we want to summarize this distribution

# The Normal Distribution

$$f(x;\mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

# The Normal Distribution: Exercises

In the module, go through the exercises in *The Normal Distribution: Exercises*

# The Normal Distribution: the log likelihood

If we take logs of the density,

$$f(x;\mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

we get this:

$$l(\mu,\sigma^2;x) = -\frac{1}{2} log(2\pi\sigma^2) - \frac{(x-\mu)^2}{2\sigma^2}$$

The likelihood for a single data point. For $\mu$ this is just a quadratic

# The Likelihood for $\mu$

The likelihood for $n$ independent samples is the product of each likelihood:

$$L(\mu, \sigma; x_1, ..., x_n) = p(x_1; \mu, \sigma)p(x_2; \mu, \sigma)...p(x_n; \mu, \sigma) = \prod_{i=1}^{n} p(x_i; \mu, \sigma)$$

This means that the log-likelihood is the sum of the likelihoods, the sum of quadratic terms:

$$\log L(\mu, \sigma; x_1, ..., x_n) = \sum_{i=1}^{n} l(x_i; \mu, \sigma) = C - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \mu)^2$$

In practice, we can calculate this with `dnorm(..., log=TRUE)`

# Our Task

Estimate the parameters of this distribution

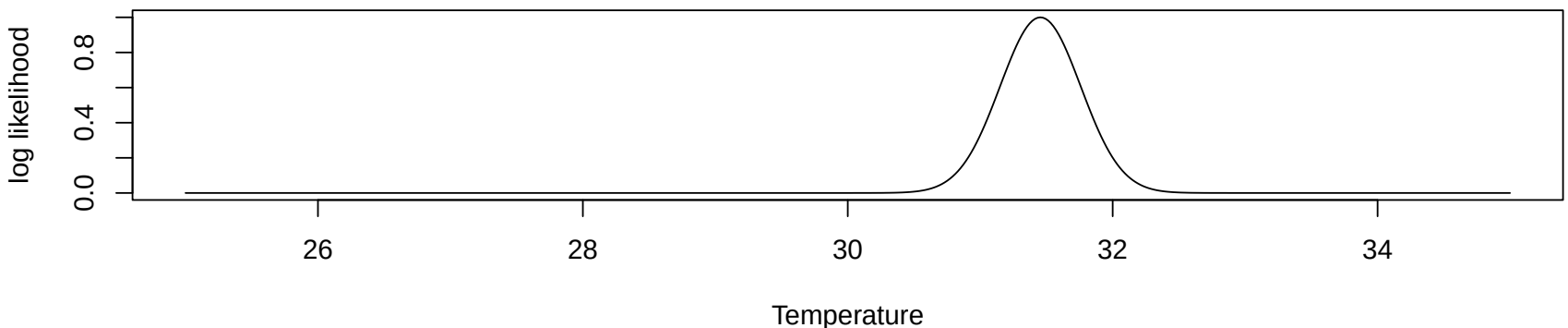
- estimate $\hat{\mu}$ and $\hat{\sigma}^2$

In practice, $\hat{\mu}$ is more important, because we will be modelling $\mu$ as a function of different effects

# Finding the Estimate

We can simulate data and calculate the likelihood for different values of the mean(we will fix the standard deviation for now)

```
CalcNormLh <- function(mu, sigma, data) {
  lhood <- sum(dnorm(data, mean=mu, sd=sigma, log=TRUE))
  lhood
}
Means <- seq(25, 35, length=500)
sdTemp <- sd(GDay$Temperature)
lhoods <- sapply(Means, CalcNormLh, sigma=sdTemp,
                 data=GDay$Temperature)
plot(Means, exp(lhoods-max(lhoods)), type="l",
     xlab="Temperature", ylab = "log likelihood")
```



# The MLE for $\mu$

We could try simulating & finding the best value, or we could try numerically maximising this. But we can get an analytic solution

(this is one reason why the normal distribution is so nice - the maths is relatively easy)

# The MLE for $\mu$

We can differentiate the log-likelihood w.r.t $\mu$, and set this to zero

$$0 = \frac{1}{2\sigma^2}\left(2\sum_{i=1}^{n} x_i - 2n\mu\right)$$

Then re-arrange, and the MLE is

$$\hat{\mu} = \frac{\sum_{i=1}^{n} x_i}{n}$$

The sample mean!

# The MLE for $\sigma^2$

This is usually less important. We are generally not interested in the standard deviation, but it is a parameter of the distribution, so it has to be estimated. What we do is differentiate w.r.t $\sigma^2$, set to zero, re-arrange, and get

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \hat{\mu})^2$$

For details, you can do it yourself or see https://www.statlect.com/fundamentals-of-statistics/normal-distribution-maximum-likelihood

# Comments

The estimate $\hat{\mu}$ is just the sample mean, and $\hat{\sigma}^2$ is just the sample variance

- the whole distribution can be summarized by these two statistics

$\hat{\sigma}^2$ has $n$ as a denominator, not $n-1$

- because we assume the MLE for $\hat{\mu}$: using (n-1) is better because it takes into account the uncertainty

# Exercises

*Confidence Intervals* is the place to be for exercises on the MLE for the mean and its confidence interval

# The distribution of $\hat{\mu}$

We can look at the distribution of $\hat{\mu}$, and (for example) estimate confidence intervals:

# The distribution of $\hat{\mu}$

If the data are normally distributed, the distribution of $\hat{\mu}$ is more accurately represented with a t-distribution (due to variance).

- we can use `dt`, `pt` etc.

The parameters:

- mean
- standard error ($=$ standard deviation$/\sqrt{n}$)
- degrees of freedom ($= n - 1$)

```
dt(32, mean(GDay$Temperature), sd(GDay$Temperature)/sqrt(le
   df=length(GDay$Temperature)-1)
```

```
## [1] 0.1690379
```

If we have enough data, this distribution look like a normal distribution

# Your work

Answer the exercises in *The distribution of* $\hat{\mu}$

and then *How the amount of data affects confidence intervals* to look at how the amount of data affects confidence intervals

# Next: something useful...

So far we have been learning about statistical inference and statistical programming. Now we can start to use this in modelling.

Next, we will start with a simple model, but put it in the context of maximum likelihood

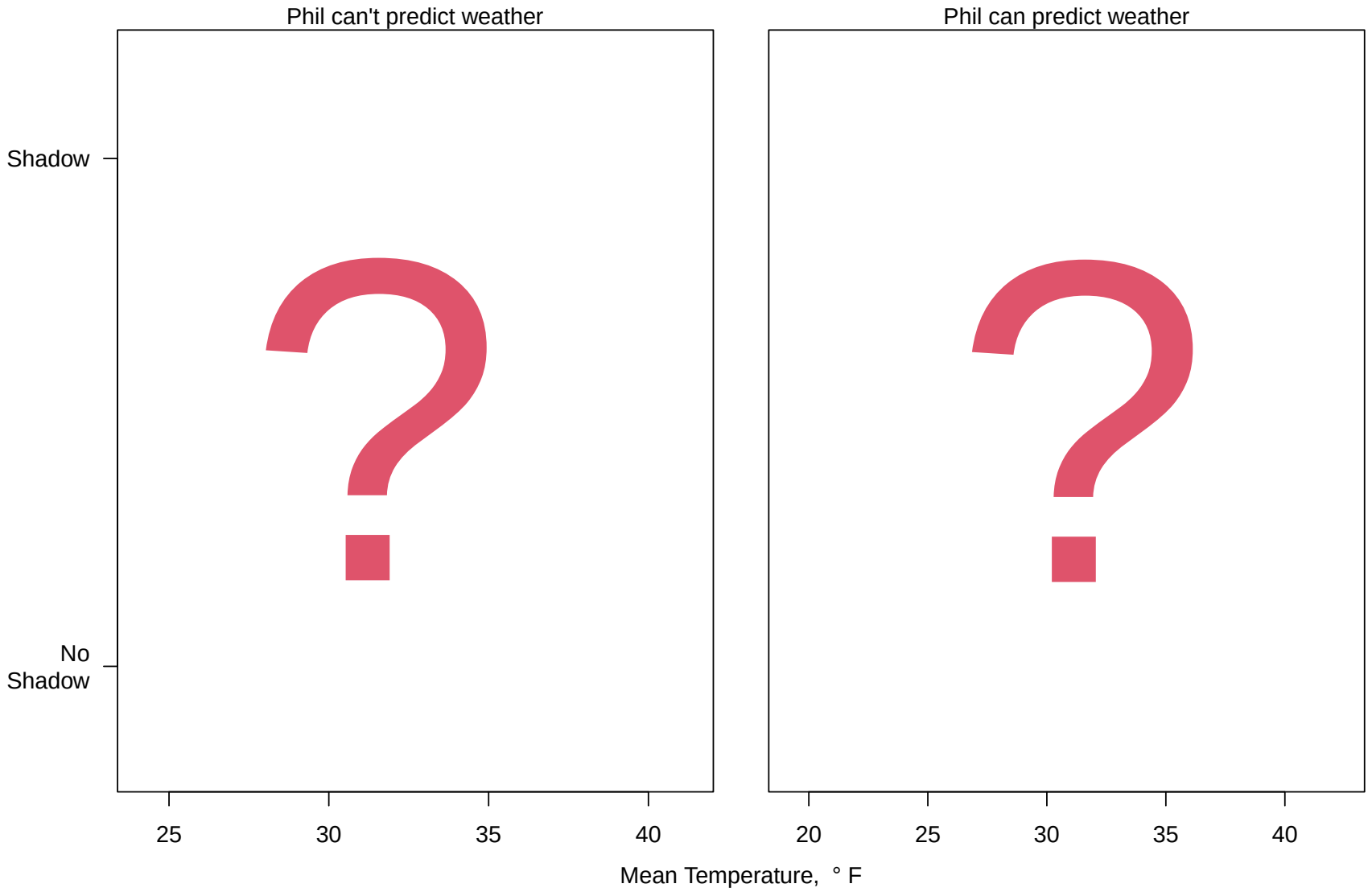
# Predicting the End of Winter

If Punxsutawney Phil sees his shadow, there will be 6 more weeks of winter

If he is good at predicting winter, we should see lower average temperatures in the 2 months after the prediction

# The Modelling

First, what would we expect if Punxsutawney Phil can predict winter, and if he cannot



# The Model

The question is about the mean temperature: is there a clear difference when Phil sees his shadow or not?

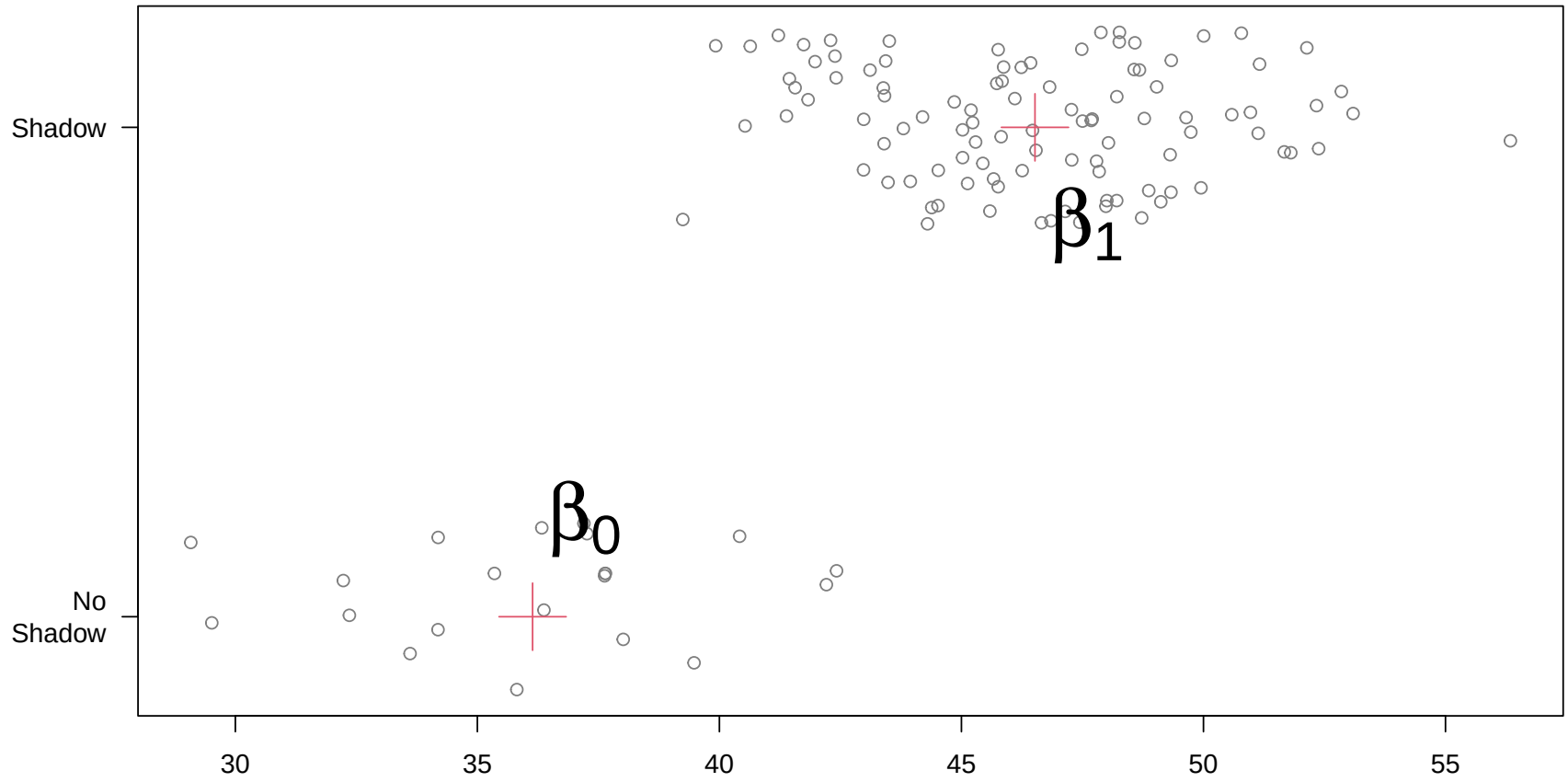
So, we have to build a model where there are different mean temperatures when he does see his shadow, and when he does not.

We can do this in a few ways. For all we assume $y_i \sim N(\mu_i, \sigma^2)$ (i.e. the data follow a normal distribution with the same variance)

We also will define a variable $X_i$: $X_i = 1$ if Phil saw his shadow (and hence predicted winter),$X_i = 0$ if he did not

# The Model - First way

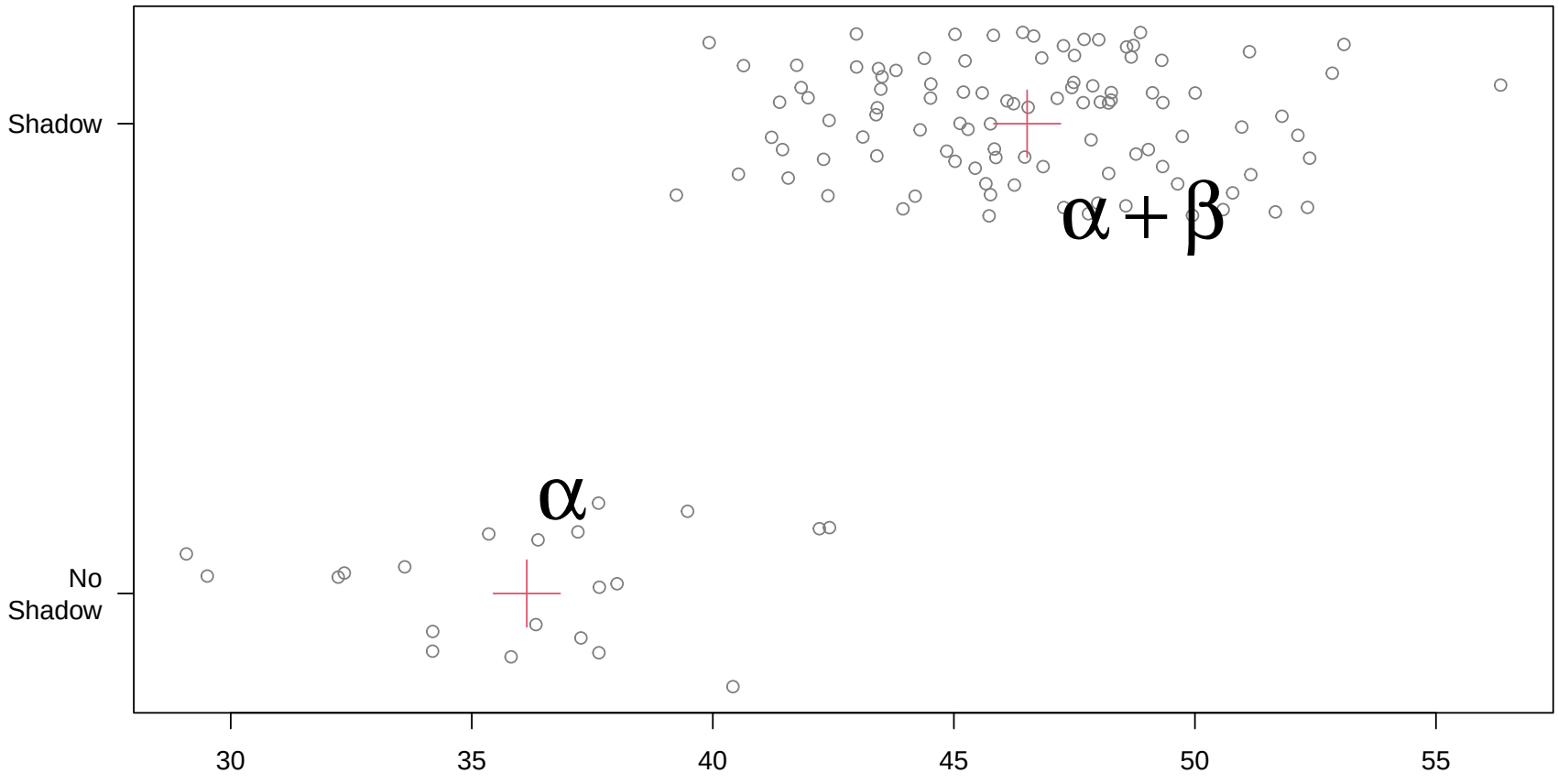
$$\mu_i = \begin{cases} \beta_0 & \text{if } X_i = 0 \\ \beta_1 & \text{if } X_i = 1 \end{cases}$$



# The Model - Second way

$$\mu_i = \begin{cases} \alpha & \text{if } X_i = 0 \\ \alpha + \beta & \text{if } X_i = 1 \end{cases}$$

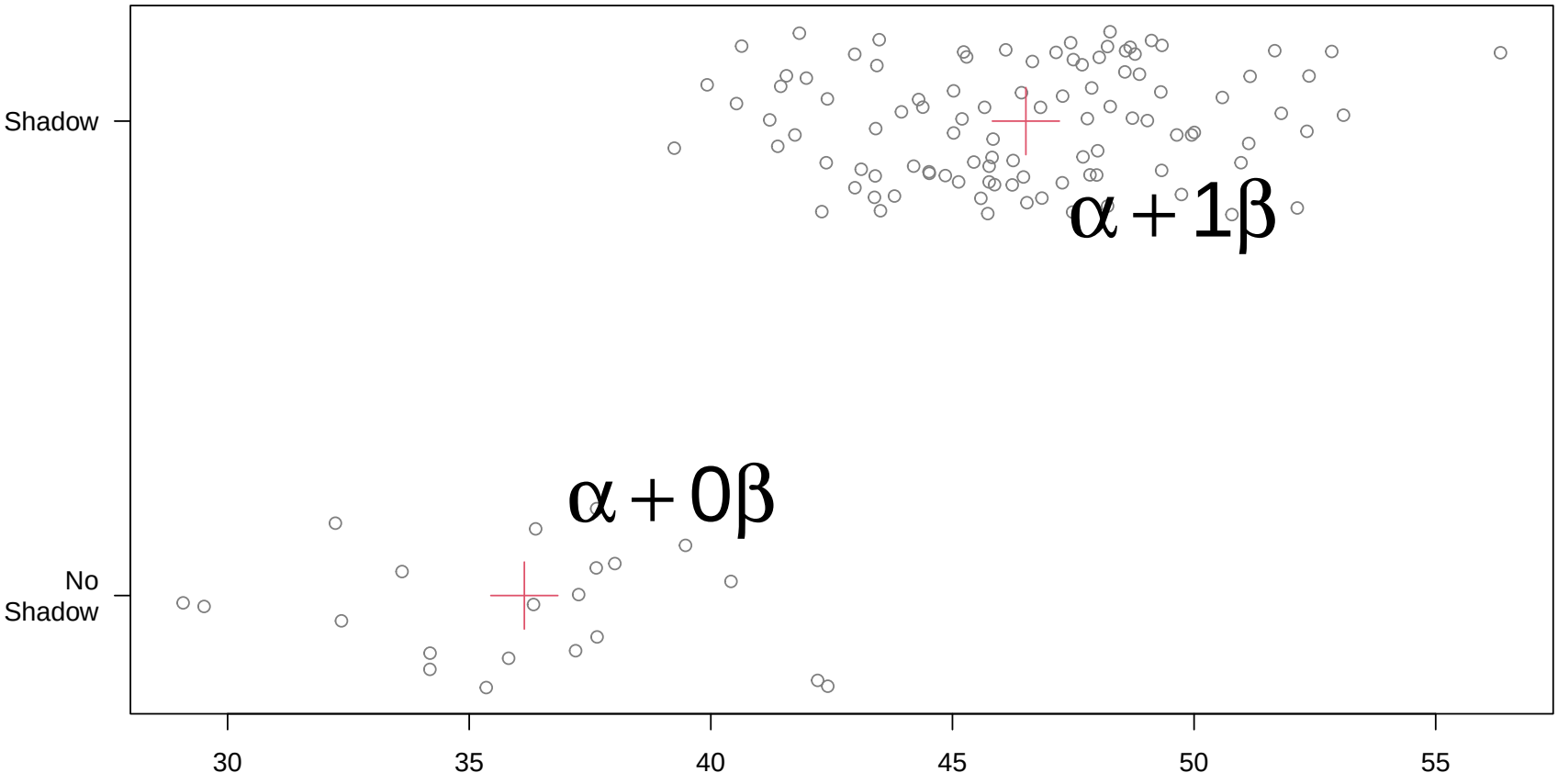
So the difference is $\beta$, and this is what we are interested in



# The Model - Third way

$$\mu_i = \alpha + \beta X_i$$

The difference is $\beta$ (because $X_i$ can only be 0 or 1). This approach is the easiest to extend to more complex models



# Calculating the Likelihood

The log-likelihood is not too difficult to write down

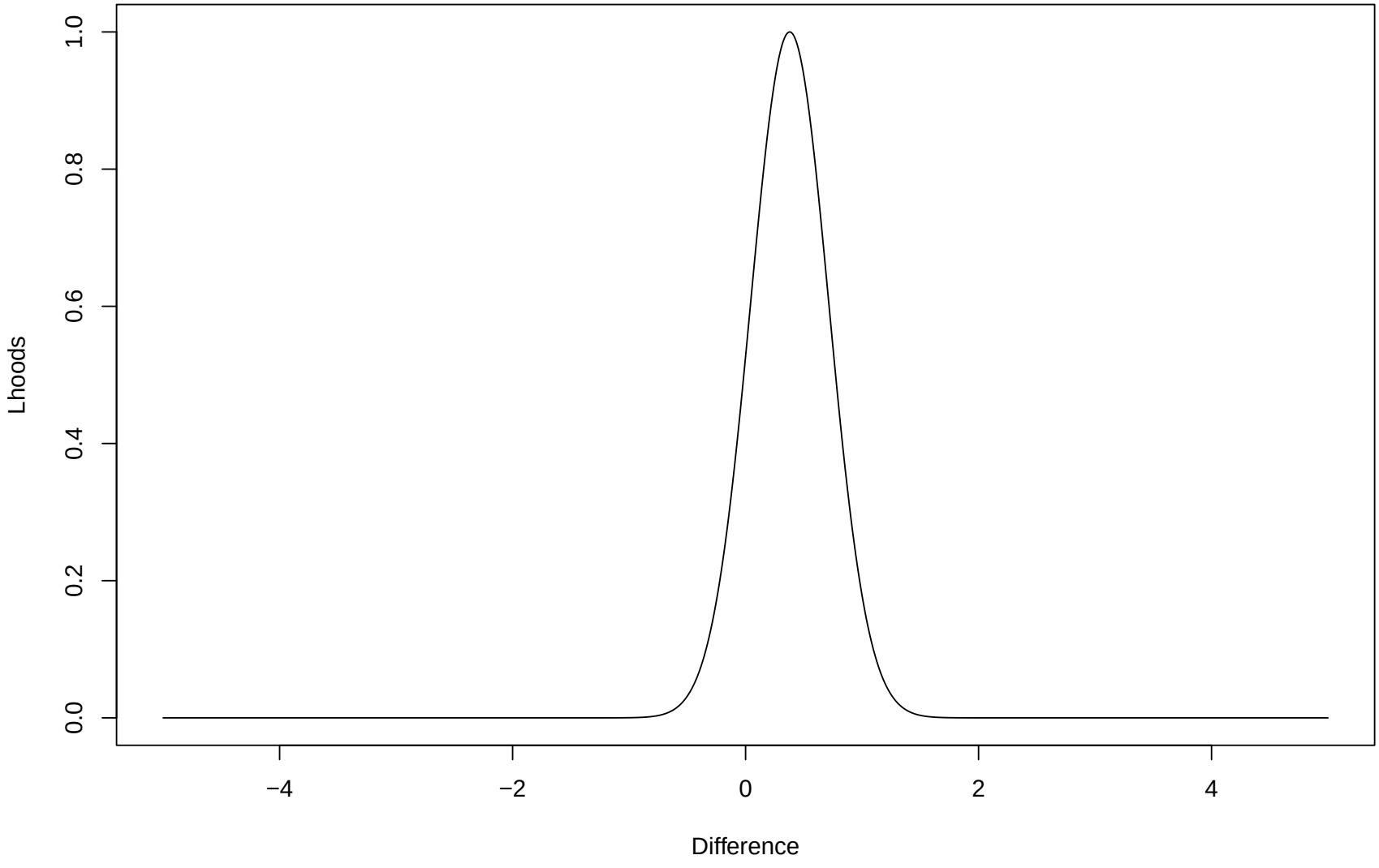
$$\log L(\mu_i, \sigma; x_1, ..., x_n) = C - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \mu_i)^2$$

where $\mu_i$ is described in the previous slide.

We now have 3 parameters (two for the means, and the standard deviation).

# Calculate the likelihood for the difference

We can look at the likelihood for the difference, using the MLEs for the other parameters



# Exercise

Do the exercise in *Calculating the Likelihood* to estimate the mle for the difference in temperatures, and the confidence interval.

Then look at the *In Practice* section on the R functions to do this, and do those exercises

Can Punxsutawney Phil predict winter?

# Exercises hand-in

- hand-in **one** file per group
  - if you're late, just hand it in anyway
- on BB under "coursework" -> exercise 1
- feedback via BB within 2 weeks
- general comments on a Friday