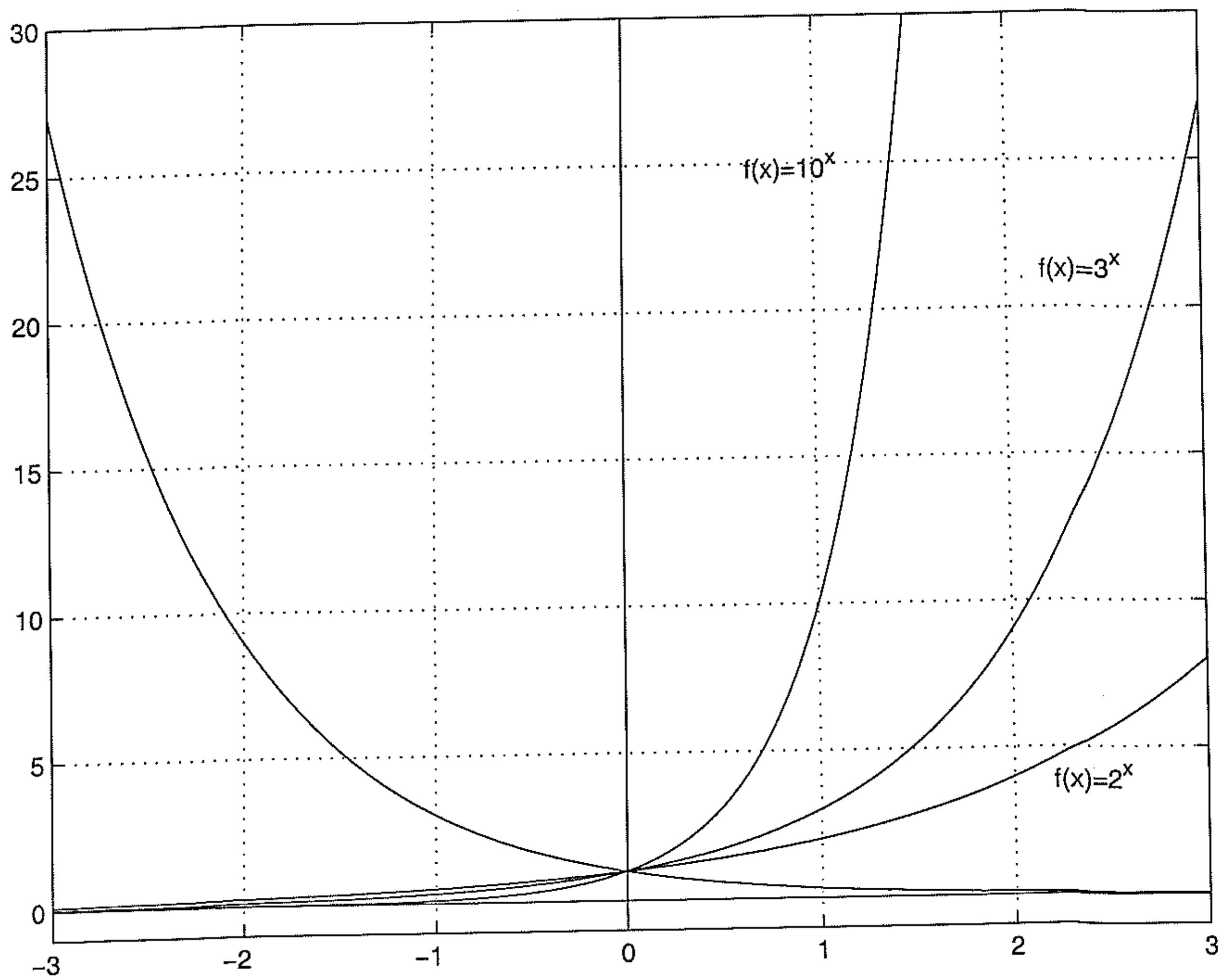
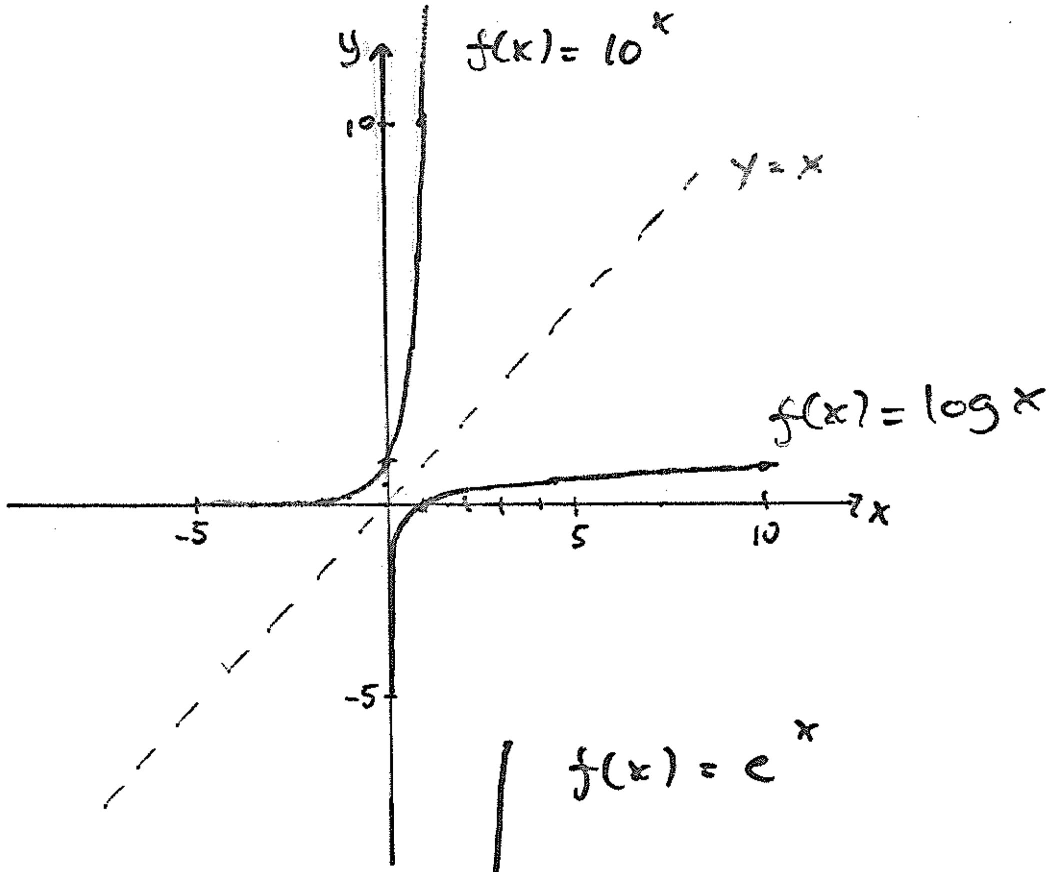
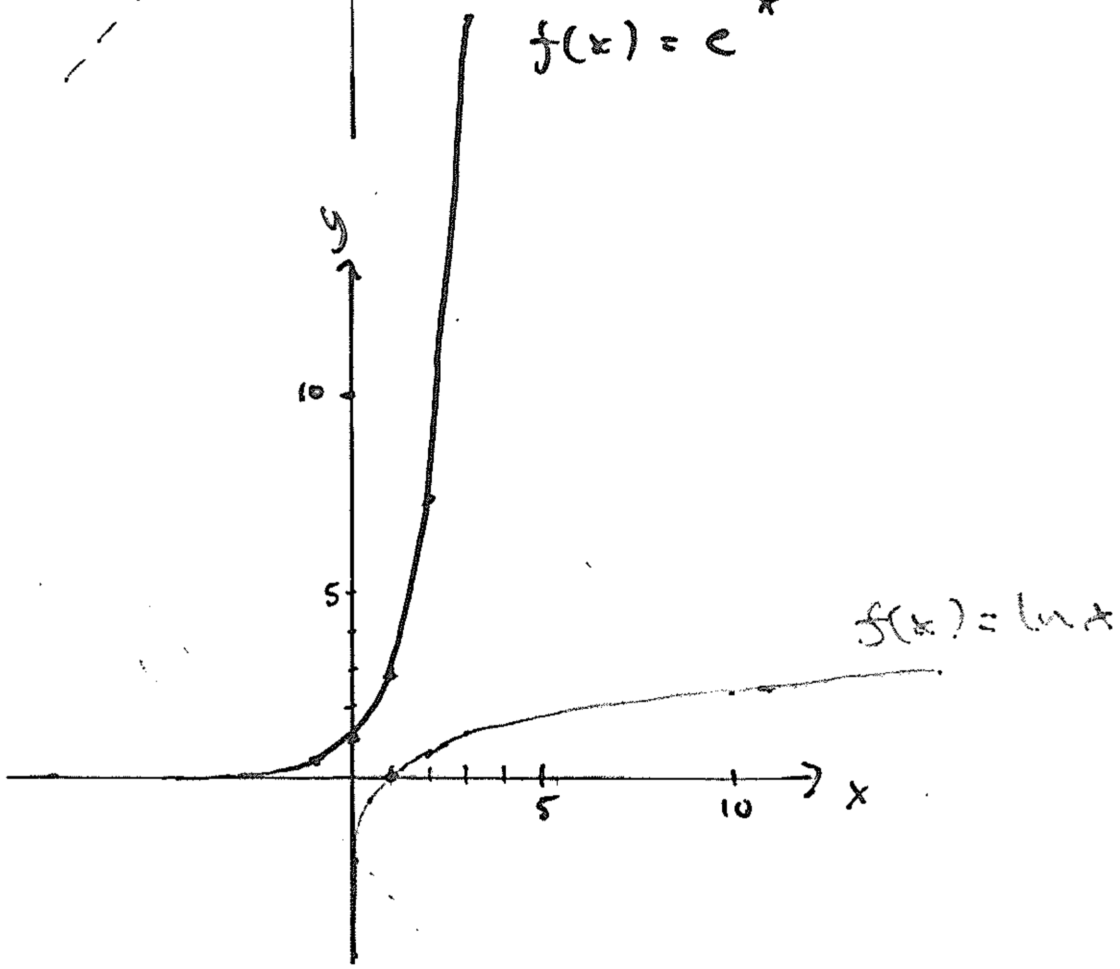


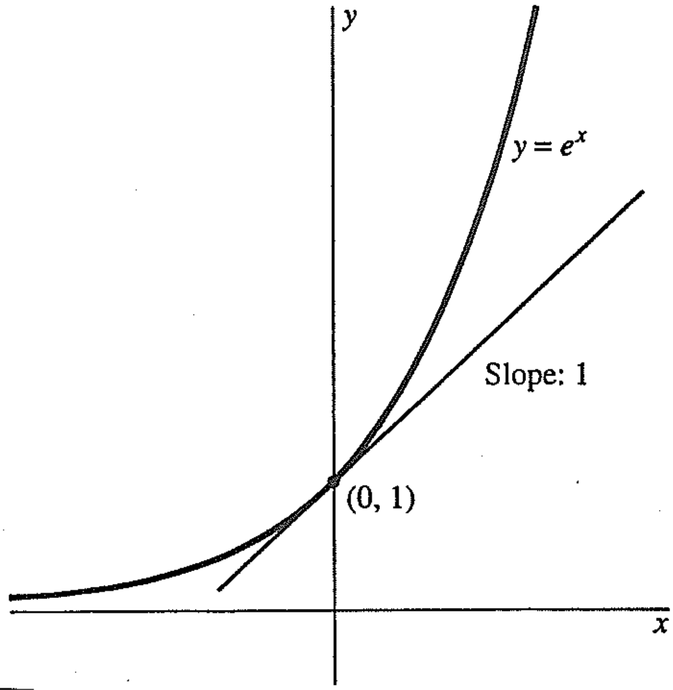
f(x)=10^x
f(x)=3^x
f(x)=2^x
30
25
20
15
10
5
0
−3
−2
−1
0
1
2
3


y
f(x) = 10^x
10
y = x
f(x) = log x
-5
5
10
x
-5


f(x) = e^x
y
10
5
f(x) = ln x
5
10
x

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

n	$\left(1+\frac{1}{n}\right)^n$
10	2.594
100	2.705
1,000	2.717
10,000	2.718
100,000	2.718



DEFINITION Inverse Functions

The two functions f and g are **inverse functions**, or are **inverses** of each other, provided that

- The range of values of each function is the domain of definition of the other, and
- The relations in (12) hold for all x in the domains of g and f, respectively.

$$f(g(x)) = x \quad \text{and} \quad g(f(x)) = x \qquad \textbf{(12)}$$

THEOREM 1 Differentiation of an Inverse Function

Suppose that the differentiable function f is defined on the open interval I and that $f'(x) > 0$ for all x in I. Then f has an inverse function g, the function g is differentiable, and

$$g'(x) = \frac{1}{f'(g(x))} \qquad \textbf{(14)}$$

for all x in the domain of g.

THEOREM 1 L'Hôpital's Rule

Suppose that the functions f and g are differentiable and that $g'(x)$ is nonzero in some neighborhood of the point a (except possibly at a itself). Suppose also that

$$\lim_{x \to a} f(x) = 0 = \lim_{x \to a} g(x).$$

Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}, \tag{2}$$

provided that the limit on the right either exists (as a finite real number) or is $+\infty$ or $-\infty$.

THEOREM 2 L'Hôpital's Rule (weak form)

Suppose that the functions f and g are differentiable at $x = a$, that

$$f(a) = 0 = g(a),$$

and that $g'(a) \neq 0$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}.$$

DEFINITION The Natural Logarithm

The **natural logarithm** $\ln x$ of the positive number x is defined to be

$$\ln x = \int_1^x \frac{1}{t}\, dt.$$

THEOREM 1 Laws of Logarithms

If x and y are positive numbers and r is a rational number, then

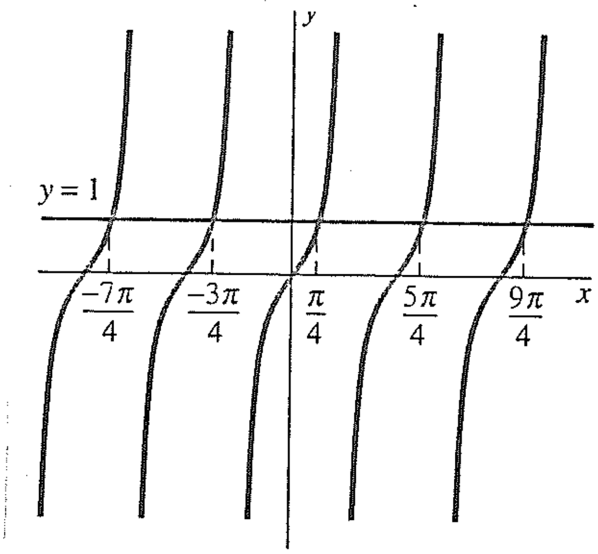
$$\ln xy = \ln x + \ln y;$$

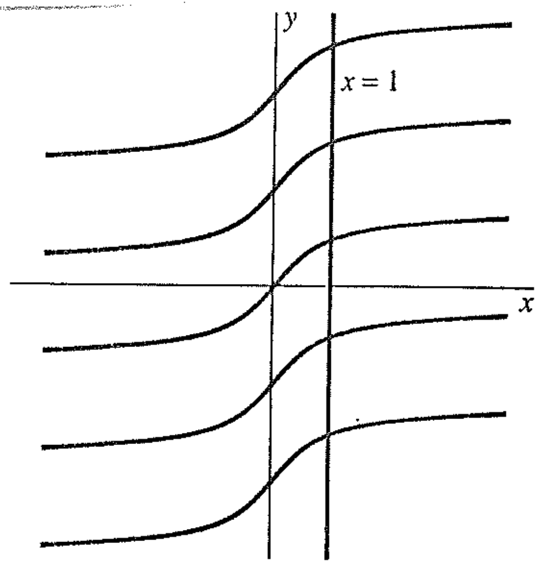
$$\ln\left(\frac{1}{x}\right) = -\ln x;$$

$$\ln\left(\frac{x}{y}\right) = \ln x - \ln y;$$

$$\ln(x^r) = r \ln x.$$

The Inverse Tangent Function



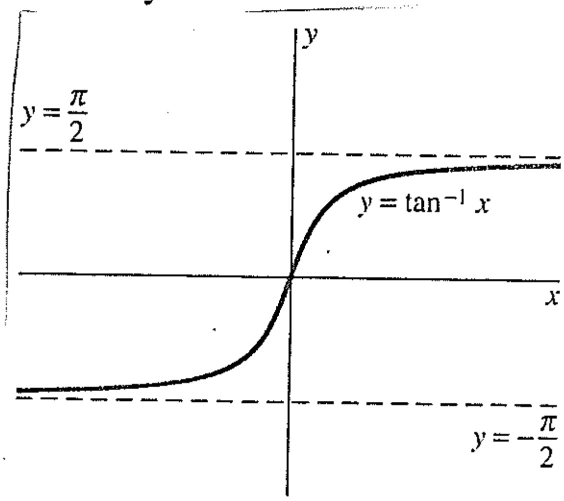


DEFINITION The Inverse Tangent Function

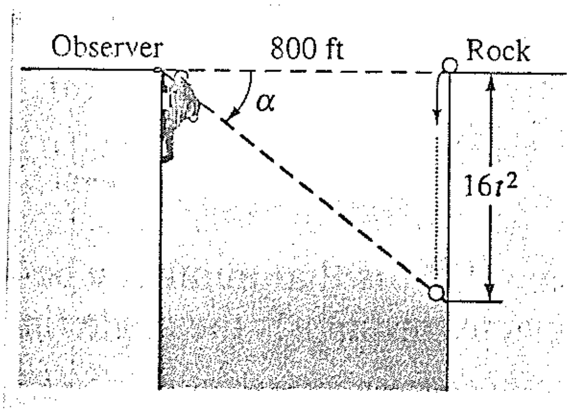
The **inverse tangent** (or **arctangent**) **function** is defined as follows:

$$y = \tan^{-1} x \quad \text{if and only if} \quad \tan y = x \quad \text{and} \quad -\pi/2 < y < \pi/2$$

where x is an arbitrary real number.



EXAMPLE 1 A mountain climber on one edge of a deep canyon 800 ft wide sees a large rock fall from the opposite edge at time $t = 0$. As he watches the rock plummet downward, his eyes first move slowly, then faster, then more slowly again. Let α be the angle of depression of his line of sight below the horizontal. At what angle α would the rock *seem* to be moving the most rapidly? That is, when would $d\alpha/dt$ be maximal?

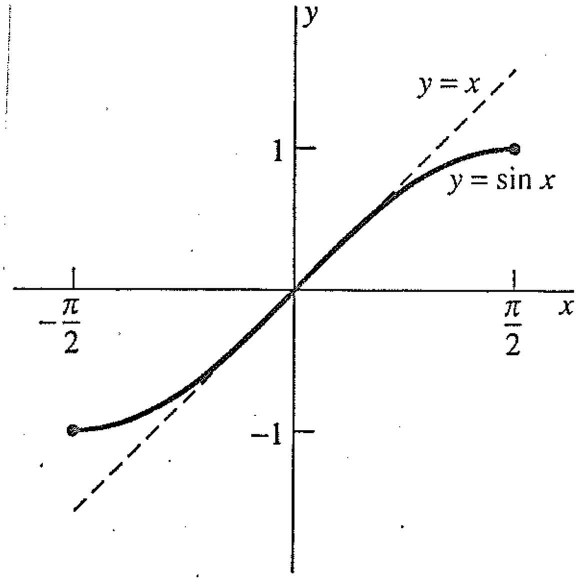


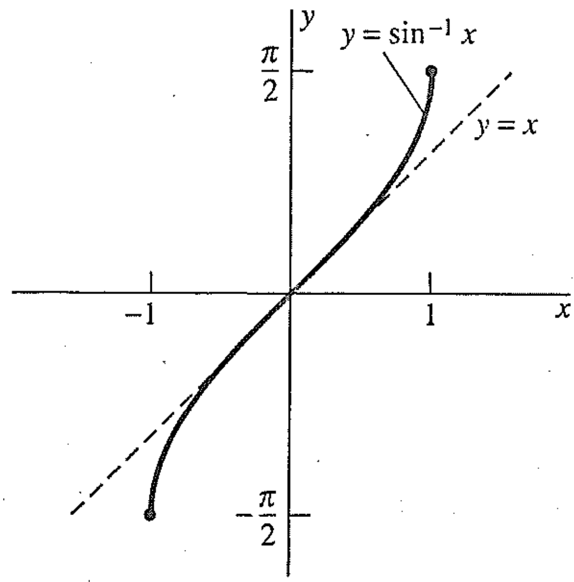
DEFINITION The Inverse Sine Function

The **inverse sine** (or **arcsine**) **function** is defined as follows:

$$y = \sin^{-1} x \quad \text{if and only if} \quad \sin y = x \quad \text{and} \quad -\pi/2 \leqq y \leqq \pi/2$$

where $-1 \leqq x \leqq 1$.





$$y = \cos^{-1} x \quad \text{if and only if} \quad \cos y = x \quad \text{and} \quad 0 \leqq y \leqq \pi$$

where $-1 \leqq x \leqq 1$.

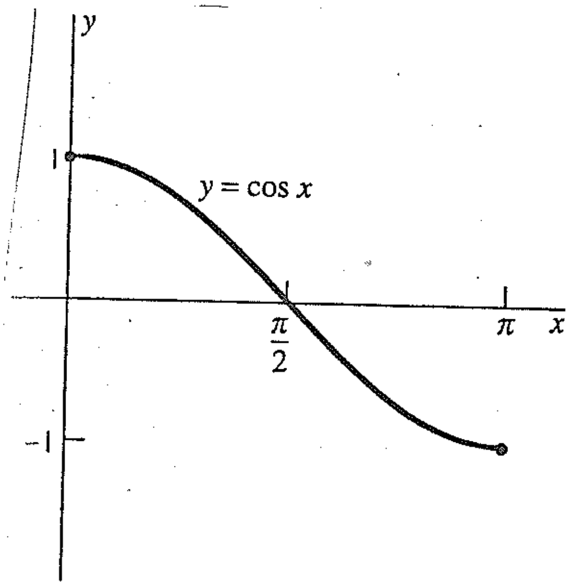


FIGURE 7.5.9 The cosine function is decreasing on the interval $0 \leqq x \leqq \pi$.

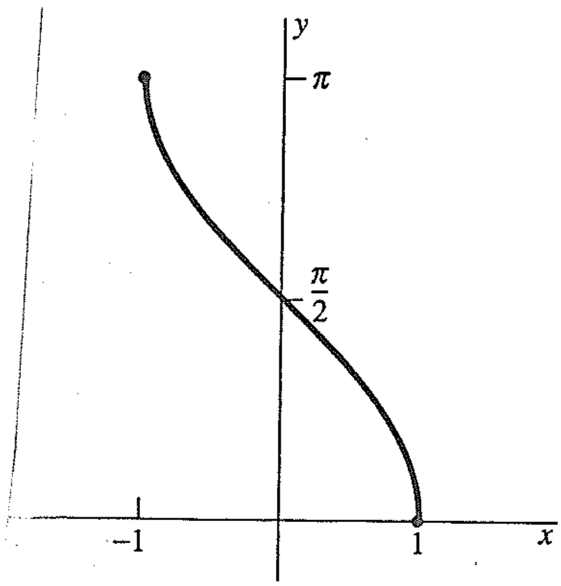


FIGURE 7.5.10 The graph $y = \cos^{-1} x$ of the arccosine function.

DEFINITION The Inverse Secant Function

The **inverse secant** (or **arcsecant**) **function** is defined as follows:

$$y = \sec^{-1} x \quad \text{if and only if} \quad \sec y = x \quad \text{and} \quad 0 \leqq y \leqq \pi$$

where $|x| \geqq 1$.

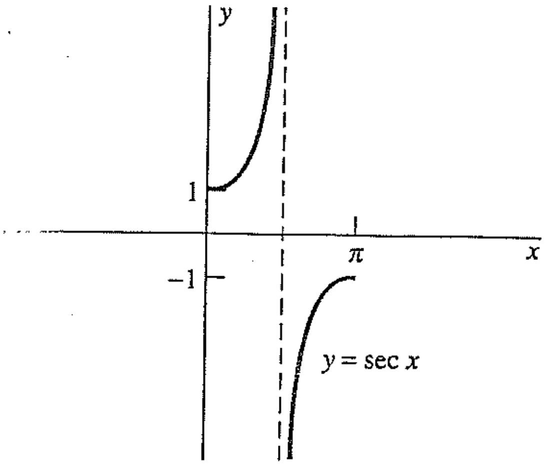


FIGURE 7.5.11 Restriction of the secant function to the union of the two intervals $[0, \pi/2)$ and $(\pi/2, \pi]$.

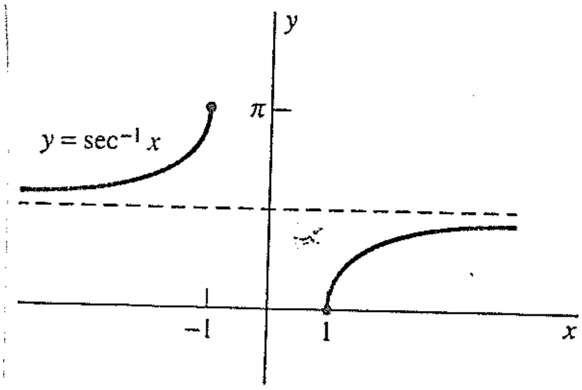


FIGURE 7.5.12 The graph of $y = \operatorname{arcsec} x = \sec^{-1} x$.

Function	Domain of Definition	Range of Values	Derivative
$\sin^{-1} x$	$-1 \leqq x \leqq 1$	$-\pi/2 \leqq y \leqq \pi/2$	$\dfrac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$-1 \leqq x \leqq 1$	$0 \leqq y \leqq \pi$	$-\dfrac{1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	$-\infty < x < +\infty$	$-\pi/2 < y < \pi/2$	$\dfrac{1}{1+x^2}$
$\cot^{-1} x$	$-\infty < x < +\infty$	$0 < y < \pi$	$-\dfrac{1}{1+x^2}$
$\sec^{-1} x$	$\lvert x\rvert \geqq 1$	$0 \leqq y < \pi/2,\ \pi/2 < y \leqq \pi$	$\dfrac{1}{\lvert x\rvert\sqrt{x^2-1}}$
$\csc^{-1} x$	$\lvert x\rvert \geqq 1$	$-\pi/2 < y < 0,\ 0 < y < \pi/2$	$-\dfrac{1}{\lvert x\rvert\sqrt{x^2-1}}$

HYPERBOLIC FUNCTIONS

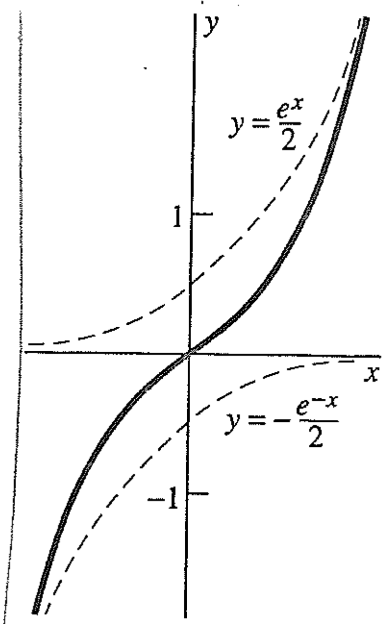
$$\cosh x = \frac{e^x + e^{-x}}{2} \quad \text{and} \quad \sinh x = \frac{e^x - e^{-x}}{2}.$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}},$$

$$\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}} \quad (x \neq 0);$$

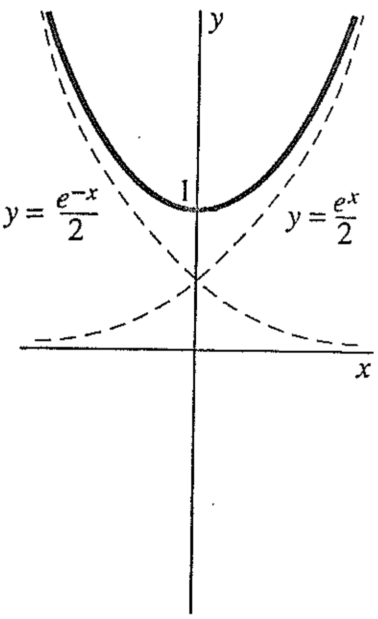
$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}},$$

$$\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}} \quad (x \neq 0).$$



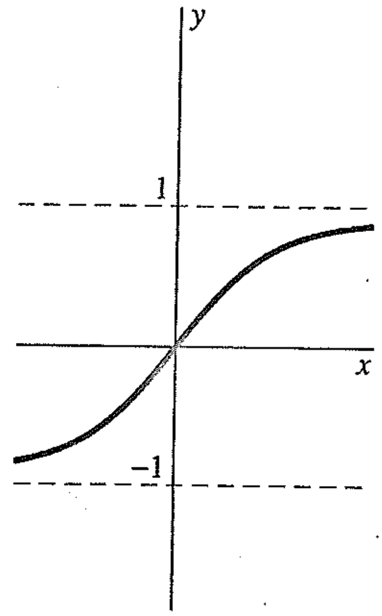
$y = \sinh x$

(a)



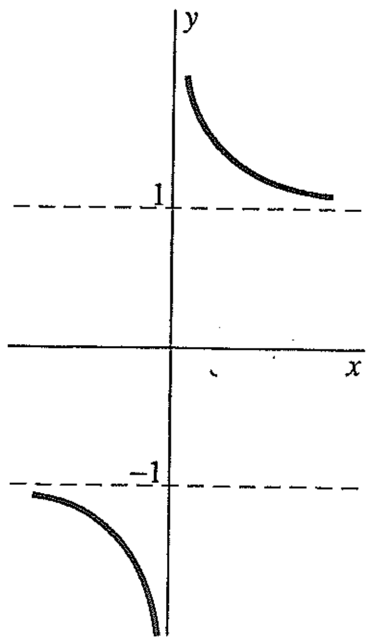
$y = \cosh x$

(b)

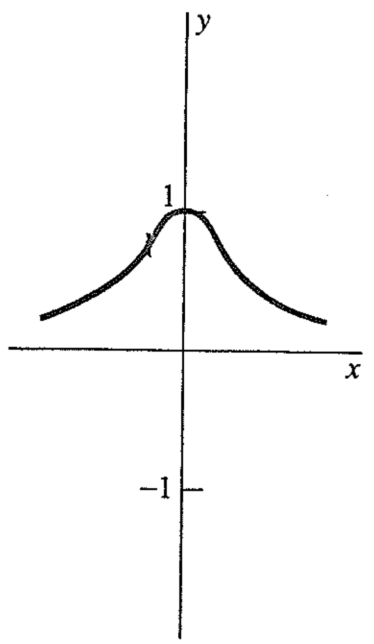


$y = \tanh x$

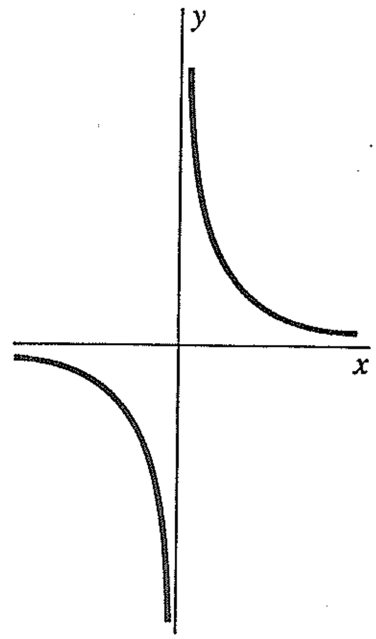
(c)



$y = \coth x$



$y = \operatorname{sech} x$



$y = \operatorname{csch} x$

$$\cosh^2 x - \sinh^2 x = 1;$$

$$1 - \tanh^2 x = \operatorname{sech}^2 x;$$

$$\coth^2 x - 1 = \operatorname{csch}^2 x;$$

$$\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y;$$

$$\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y;$$

$$\sinh 2x = 2\sinh x \cosh x;$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x;$$

$$\cosh^2 x = \tfrac{1}{2}(\cosh 2x + 1);$$

$$\sinh^2 x = \tfrac{1}{2}(\cosh 2x - 1).$$

$$D_x \sinh u = (\cosh u)\,\frac{du}{dx},$$

$$D_x \tanh u = (\operatorname{sech}^2 u)\,\frac{du}{dx},$$

$$D_x \coth u = (-\operatorname{csch}^2 u)\,\frac{du}{dx},$$

$$D_x \operatorname{sech} u = (-\operatorname{sech} u \tanh u)\,\frac{du}{dx},$$

$$D_x \operatorname{csch} u = (-\operatorname{csch} u \coth u)\,\frac{du}{dx}.$$

$$\sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right) \qquad \text{for all } x;$$

$$\cosh^{-1} x = \ln\left(x + \sqrt{x^2 - 1}\right) \qquad \text{for all } x \geqq 1;$$

$$\tanh^{-1} x = \tfrac{1}{2} \ln\left(\frac{1 + x}{1 - x}\right) \qquad \text{for } |x| < 1;$$

$$\coth^{-1} x = \tfrac{1}{2} \ln\left(\frac{x + 1}{x - 1}\right) \qquad \text{for } |x| > 1;$$

$$\operatorname{sech}^{-1} x = \ln\left(\frac{1 + \sqrt{1 - x^2}}{x}\right) \qquad \text{if } 0 < x \leqq 1;$$

$$\operatorname{csch}^{-1} x = \ln\left(\frac{1}{x} + \frac{\sqrt{1 + x^2}}{|x|}\right) \qquad \text{if } x \neq 0.$$

Derivatives of Inverse Hyperbolic Functions

$$D_x \sinh^{-1} x = \frac{1}{\sqrt{1+x^2}},$$

$$D_x \cosh^{-1} x = \frac{1}{\sqrt{x^2-1}},$$

$$D_x \tanh^{-1} x = \frac{1}{1-x^2},$$

$$D_x \coth^{-1} x = \frac{1}{1-x^2},$$

$$D_x \operatorname{sech}^{-1} x = -\frac{1}{x\sqrt{1-x^2}},$$

$$D_x \operatorname{csch}^{-1} x = -\frac{1}{|x|\sqrt{1+x^2}}.$$

$$\int \frac{du}{\sqrt{u^2+1}} = \sinh^{-1} u + C,$$

$$\int \frac{du}{\sqrt{u^2-1}} = \cosh^{-1} u + C,$$

$$\int \frac{du}{1-u^2} = \tanh^{-1} u + C \quad \text{if } |u| < 1,$$

$$\int \frac{du}{1-u^2} = \coth^{-1} u + C \quad \text{if } |u| > 1,$$

$$\int \frac{du}{1-u^2} = \tfrac{1}{2} \ln \left| \frac{1+u}{1-u} \right| + C,$$

$$\int \frac{du}{u\sqrt{1-u^2}} = -\operatorname{sech}^{-1} |u| + C,$$

$$\int \frac{du}{u\sqrt{1+u^2}} = -\operatorname{csch}^{-1} |u| + C.$$