DEFINITION Limit of a Sequence

We say that the sequence $\{a_n\}$ **converges** to the real number L, or has the **limit** L, and we write

$$\lim_{n\to\infty} a_n = L, \tag{4}$$

provided that a_n can be made as close to L as we please merely by choosing n to be sufficiently large. That is, given any number $\epsilon > 0$, there exists an integer N such that

$$|a_n - L| < \epsilon \quad \text{for all } n \geqq N.$$

If the sequence $\{a_n\}$ does *not* converge, then we say that $\{a_n\}$ **diverges.**

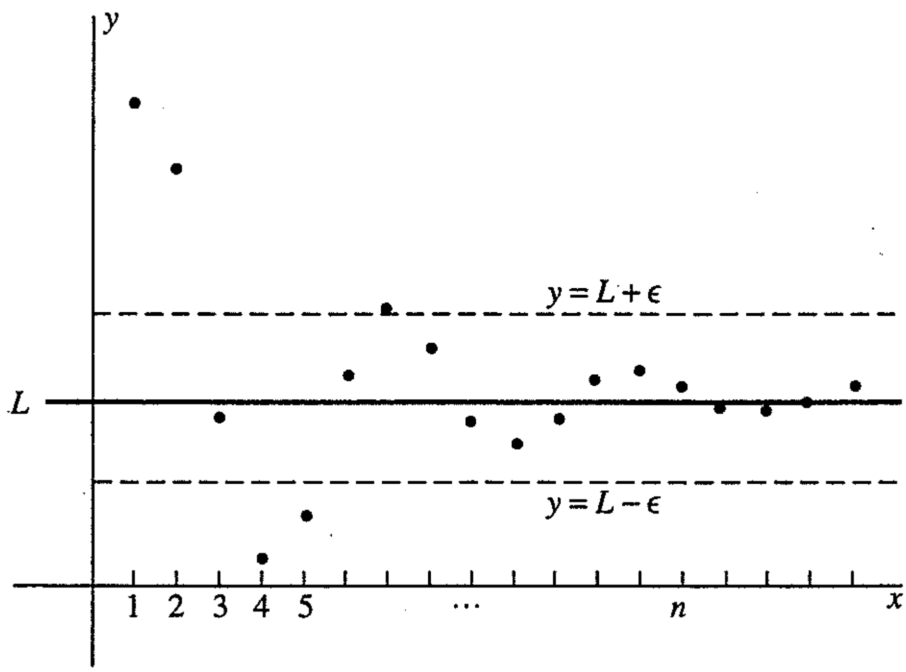


FIGURE 11.2.1 The point (n, a_n) approaches the line $y = L$ as $n \to +\infty$.

THEOREM 1 Limit Laws for Sequences

If the limits

$$\lim_{n\to\infty} a_n = A \quad \text{and} \quad \lim_{n\to\infty} b_n = B$$

exist (so A and B are real numbers), then

1. $\displaystyle\lim_{n\to\infty} ca_n = cA$ (c any real number);
2. $\displaystyle\lim_{n\to\infty} (a_n + b_n) = A + B$;
3. $\displaystyle\lim_{n\to\infty} a_n b_n = AB$;
4. $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B}$.

THEOREM 2 Substitution Law for Sequences

If $\lim_{n\to\infty} a_n = A$ and the function f is continuous at $x = A$, then

$$\lim_{n\to\infty} f(a_n) = f(A).$$

THEOREM 3 Squeeze Law for Sequences

If $a_n \leqq b_n \leqq c_n$ for all n and

$$\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n,$$

then $\lim_{n\to\infty} b_n = L$ as well.

THEOREM 4 Limits of Functions and Sequences

If $a_n = f(n)$ for each positive integer n, then

$$\lim_{x\to\infty} f(x) = L \quad \text{implies that} \quad \lim_{n\to\infty} a_n = L.$$

The sequence $\{a_n\}_1^\infty$ is said to be **increasing** if

$$a_1 \leqq a_2 \leqq a_3 \leqq \cdots \leqq a_n \leqq \cdots$$

and **decreasing** if

$$a_1 \geqq a_2 \geqq a_3 \geqq \cdots \geqq a_n \geqq \cdots.$$

The sequence $\{a_n\}$ is **monotonic** if it is either increasing or decreasing. The sequence $\{a_n\}$ is **bounded** if there is a number M such that $|a_n| \leqq M$ for all n.

Bounded Monotonic Sequence Property

Every bounded monotonic infinite sequence converges—that is, has a finite limit.

Oppgave:

UNDERSØK OM FØLGEN

$$a_1 = \sqrt{2}$$

$$a_{n+1} = \sqrt{2 + a_n}, \quad n \geq 1$$

KONVERGERER, OG FINN EVENTUELT GRENSEN.

DEFINITION The Sum of an Infinite Series

We say that the infinite series

$$\sum_{n=1}^{\infty} a_n \quad \textbf{converges} \text{ (or is } \textbf{convergent}\text{)}$$

with **sum** S provided that the limit of its sequence of partial sums,

$$S = \lim_{n\to\infty} S_n,$$

exists (and is finite). Otherwise we say that the series **diverges** (or is **divergent**). If a series diverges, then it has no sum.

Thus the sum of an infinite series is a limit of finite sums,

$$S = \sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} \sum_{n=1}^{N} a_n,$$

provided that this limit exists.

DEFINITION Geometric Series

The series $\sum_{n=0}^{\infty} a_n$ is said to be a **geometric series** if each term after the first is a fixed multiple of the term immediately before it. That is, there is a number r, called the **ratio** of the series, such that

$$a_{n+1} = r a_n \quad \text{for all } n \geqq 0.$$

THEOREM 1 The Sum of a Geometric Series

If $|r| < 1$, then the geometric series in Eq. (5) converges, and its sum is

$$S = \sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}.$$

If $|r| \geqq 1$ and $a \neq 0$, then the geometric series diverges.

THEOREM 2 Termwise Addition and Multiplication

If the series $A = \sum a_n$ and $B = \sum b_n$ converge to the indicated sums and c is a constant, then the series $\sum (a_n + b_n)$ and $\sum c a_n$ also converge, with sums

1. $\sum (a_n + b_n) = A + B$;
2. $\sum c a_n = cA$.

THEOREM 3 The *n*th-Term Test for Divergence

If either

$$\lim_{n \to \infty} a_n \neq 0$$

or this limit does not exist, then the infinite series $\sum a_n$ diverges.

THEOREM 4

The harmonic series diverges.

THEOREM 5 Series that Are Eventually the Same

If there exists a positive integer k such that $a_n = b_n$ for all $n > k$, then the series $\sum a_n$ and $\sum b_n$ either both converge or both diverge.

THEOREM 1 The nth-Degree Taylor Polynomial

Suppose that the first n derivatives of the function $f(x)$ exist at $x = a$. Let $P_n(x)$ be the nth-degree polynomial

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

$$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

THEOREM 2 Taylor's Formula

Suppose that the $(n+1)$th derivative of the function f exists on an interval containing the points a and b. Then

$$f(b) = f(a) + f'(a)(b-a) + \frac{f''(a)}{2!}(b-a)^2$$

$$+ \frac{f^{(3)}(a)}{3!}(b-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(b-a)^n + \frac{f^{(n+1)}(z)}{(n+1)!}(b-a)^{n+1}$$

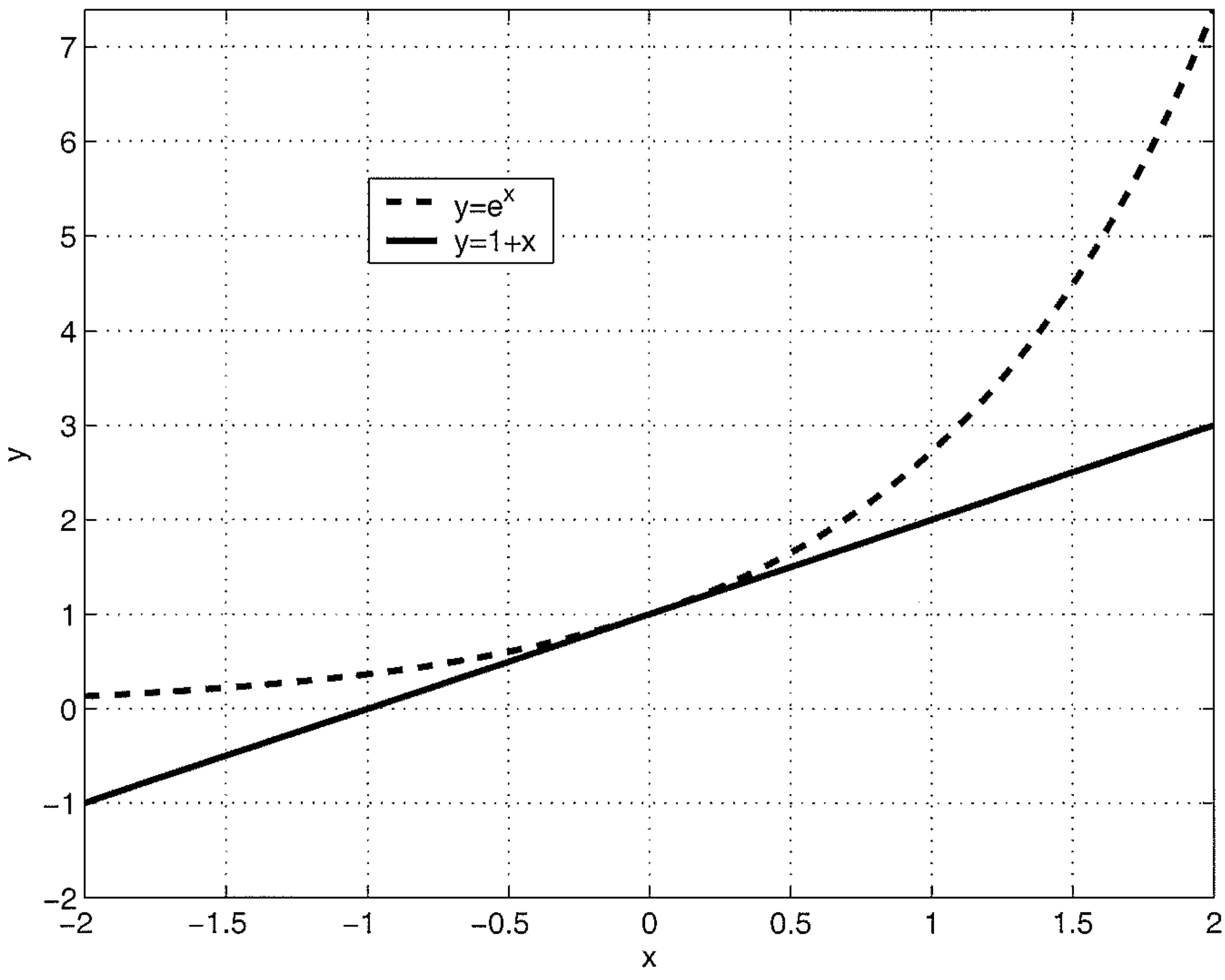
for some number z between a and b.

THE MEAN VALUE THEOREM

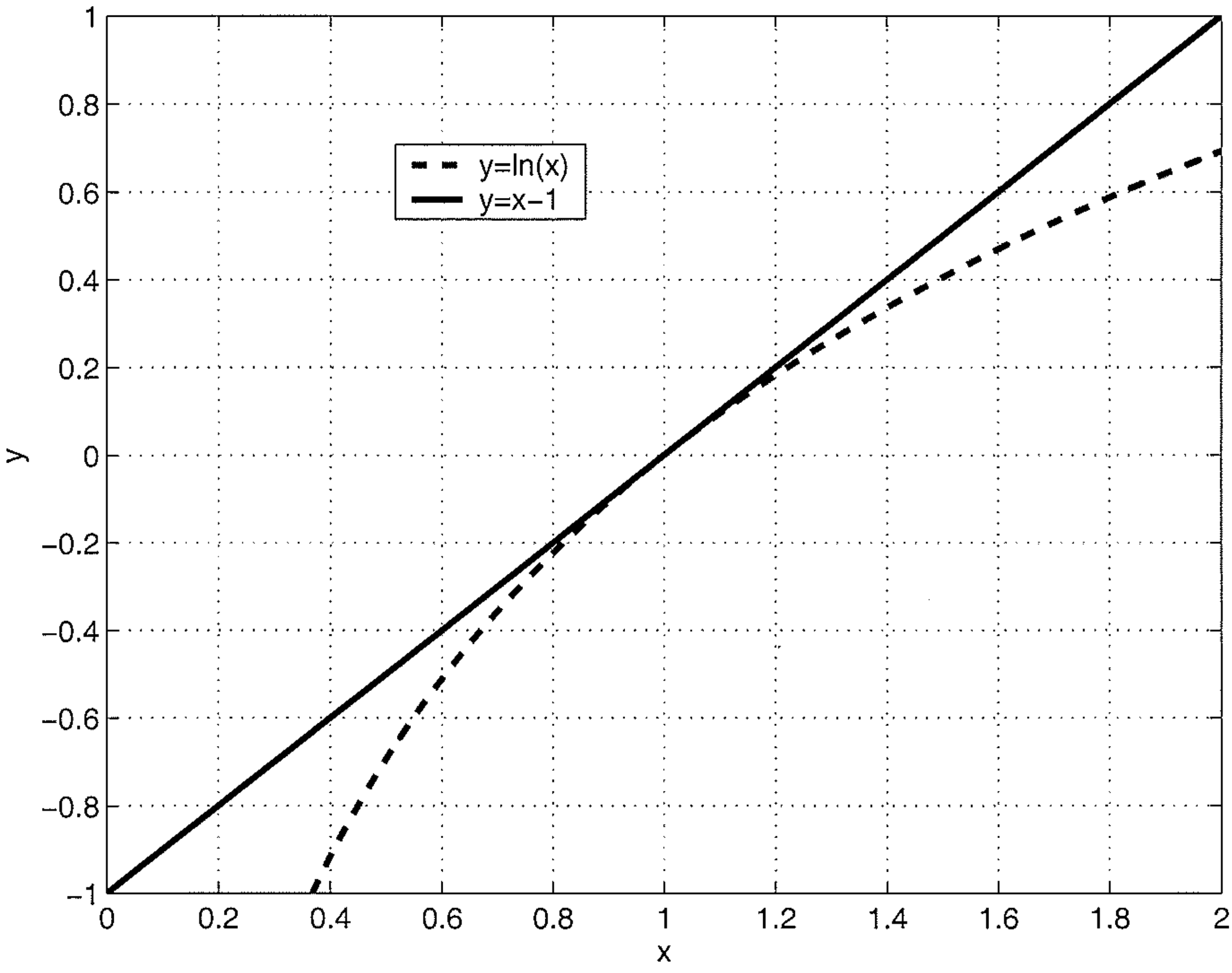
Suppose that the function f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). Then

$$f(b) - f(a) = f'(c) \cdot (b-a)$$

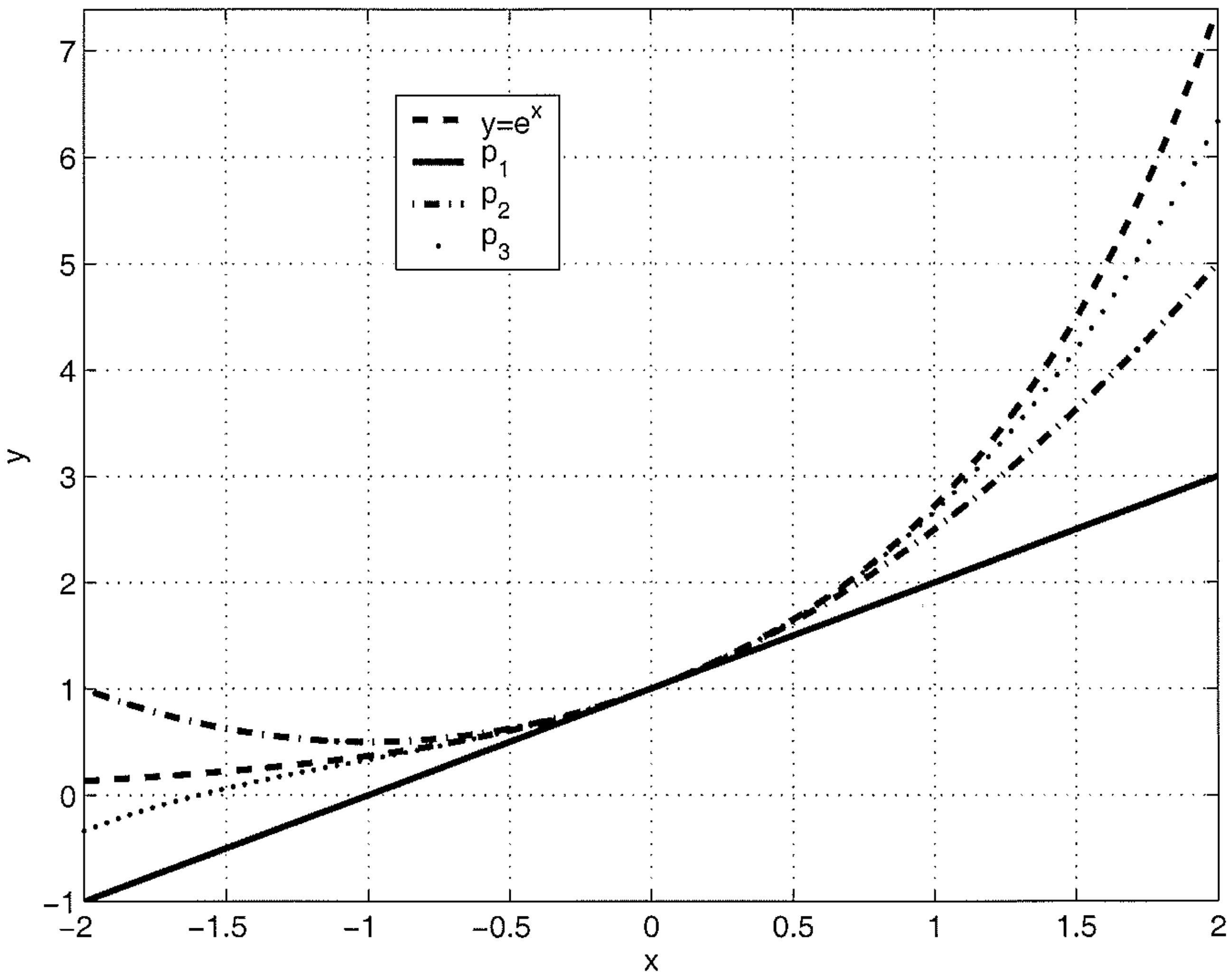
for some number c in (a, b).



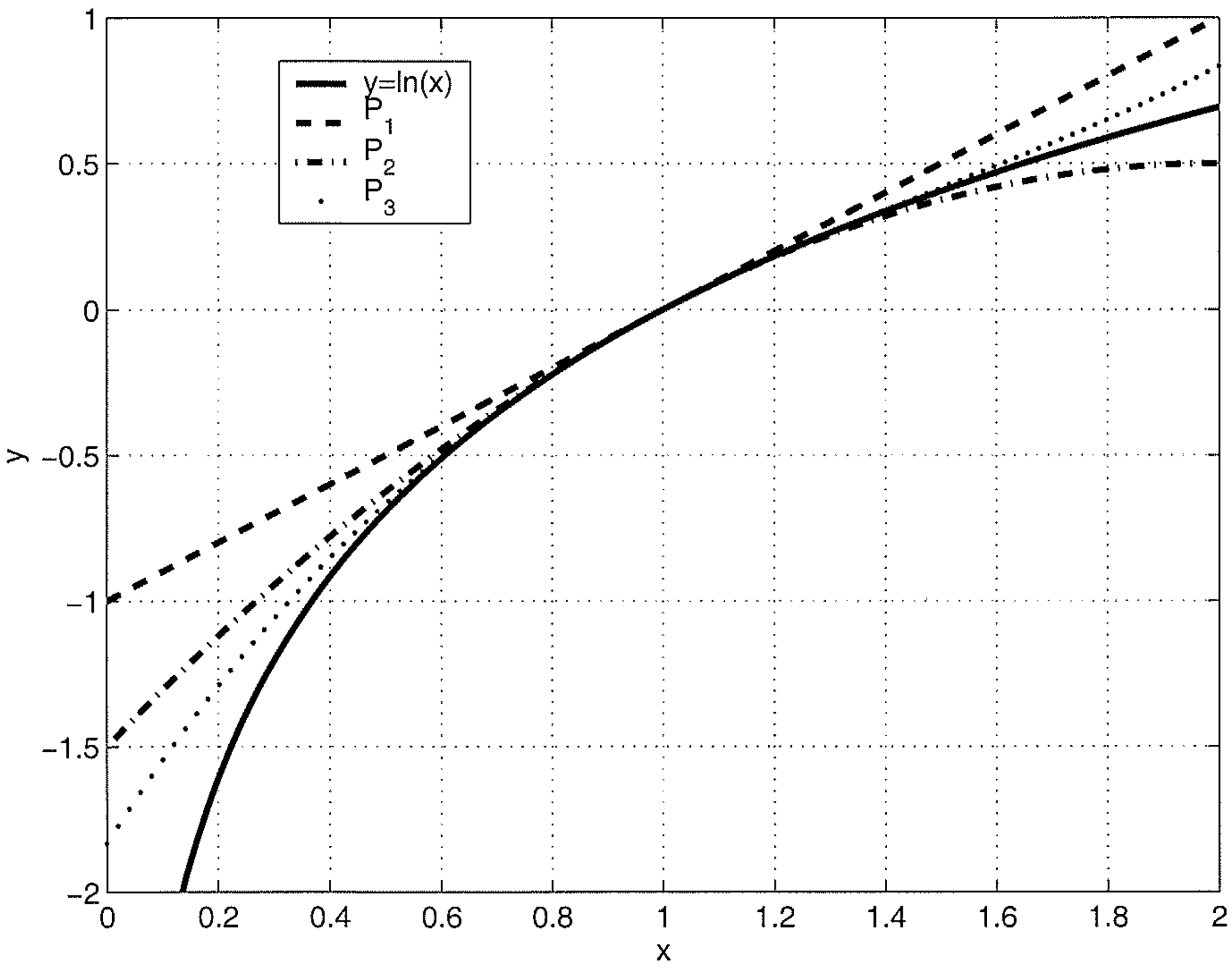
y=e^x
y=1+x
7
6
5
4
3
2
1
0
−1
−2
y
−2
−1.5
−1
−0.5
0
0.5
1
1.5
2
x



y=ln(x)
y=x−1
y
x
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2



y=e^x
p1
p2
p3
7
6
5
4
3
2
1
0
−1
−2
−1.5
−1
−0.5
0
0.5
1
1.5
2
x
y



y=ln(x)
P1
P2
P3
1
0.5
0
−0.5
−1
−1.5
−2
y
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
x