THEOREM 1 Convergence of Power Series

If $\sum a_n x^n$ is a power series, then either

1. The series converges absolutely for all x, or
2. The series converges only when $x = 0$, or
3. There exists a number $R > 0$ such that $\sum a_n x^n$ converges absolutely if $|x| < R$ and diverges if $|x| > R$.

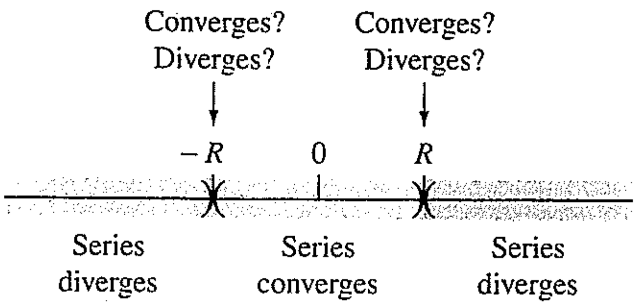


FIGURE 11.8.1 The interval of convergence if $0 < R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| < \infty$.

$$\sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots,$$

1. The series in Eq. (9) converges absolutely for all x, or
2. The series converges only when $x - c = 0$—that is, when $x = c$—or
3. There exists a number $R > 0$ such that the series in Eq. (9) converges absolutely if $|x - c| < R$ and diverges if $|x - c| > R$.

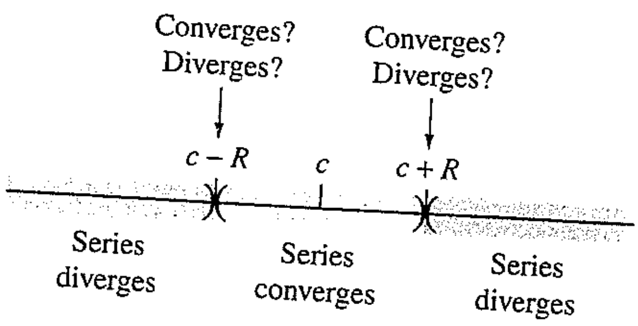


FIGURE 11.8.4 The interval of convergence of $\sum_{n=0}^{\infty} a_n(x-c)^n$.

Oppgave 5

a) Bestem konvergensradien for potensrekken

$$\sum_{n=0}^{\infty} \frac{x^n}{\sqrt{4n+1}},$$

og avgjør om rekken konvergerer i endepunktene av konvergensintervallet.

b) La S betegne summen av rekken i **a)** når $x = -1/4$. Finn en tilnærmet verdi L for S slik at $|S - L| \leq 10^{-3}$.

THEOREM 2 Taylor Series Representations
Suppose that the function f has derivatives of all orders on some interval containing a and also that

$$\lim_{n\to\infty} R_n(x) = 0$$

for each x in that interval. Then

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

for each x in the interval.

THEOREM 3 Termwise Differentiation and Integration
Suppose that the function f has a power series representation

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

with nonzero radius of convergence R. Then f is differentiable on $(-R, R)$ and

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots. \quad \textbf{(17)}$$

Also,

$$\int_0^x f(t)\,dt = \sum_{n=0}^{\infty} \frac{a_n x^{n+1}}{n+1} = a_0 x + \tfrac{1}{2}a_1 x^2 + \tfrac{1}{3}a_2 x^3 + \cdots \quad \textbf{(18)}$$

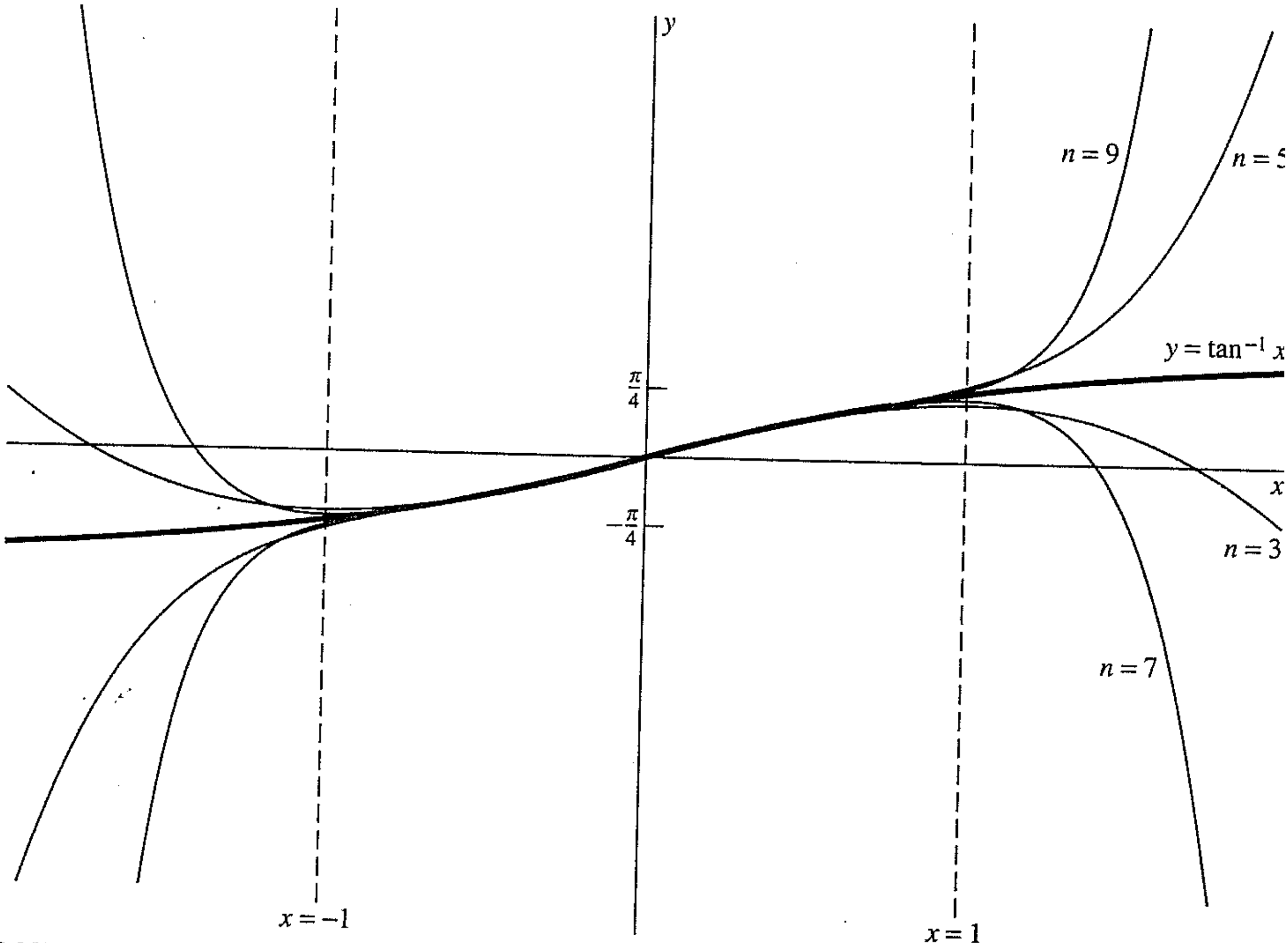
for each x in $(-R, R)$. Moreover, the power series in Eqs. (17) and (18) have the same radius of convergence R.

$$\sum_{k=1}^{\infty} (-1)^{k+1} a_k = (a_1 - a_2 + a_3 - \cdots \pm a_n) + E,$$

REMARK Suppose that we had been asked in advance to approximate $\sqrt{105}$ accurate to five decimal places. A convenient way to do this is to continue writing terms of the series until it is clear that they have become too small in magnitude to affect the fifth decimal place. A good rule of thumb is to use two more decimal places in the computations than are required in the final answer. Thus we use seven decimal places in this case and get

$$\tan^{-1} x = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{2n-1}}{2n-1} = x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - \tfrac{1}{7}x^7 + \tfrac{1}{9}x^9 - \cdots$$

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots + \frac{(-1)^{n+1}}{n}x^n + \cdots$$



THEOREM 1 Adding and Multiplying Power Series

Let $\sum a_n x^n$ and $\sum b_n x^n$ be power series with nonzero radii of convergence. Then

$$\sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} (a_n + b_n) x^n \tag{4}$$

and

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} c_n x^n$$

$$= a_0 b_0 + (a_0 b_1 + a_1 b_0)x + (a_0 b_2 + a_1 b_1 + a_2 b_0)x^2 + \cdots, \tag{5}$$

where

$$c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_{n-1} b_1 + a_n b_0. \tag{6}$$

The series in Eqs. (4) and (5) converge for any x that lies interior to the intervals of convergence of both $\sum a_n x^n$ and $\sum b_n x^n$.

FINN $\int_{-\pi/2}^{\pi/2} \frac{\tan^{-1} x}{x} \, dx$ MED

2 DESIMALERS NØYAKTIGHET

EXAMPLE 3 Assume that the tangent function has a power series representation $\tan x = \sum a_n x^n$ (it does). Use the Maclaurin series for $\sin x$ and $\cos x$ to find a_0, a_1, a_2, and a_3.

$$\tan x = x + \tfrac{1}{3}x^3 + \tfrac{2}{15}x^5 + \tfrac{17}{315}x^7 + \tfrac{62}{2835}x^9 + \tfrac{1382}{155.925}x^{11} + \cdots$$

$$\lim_{x \to 0} \frac{x - \sin x}{x^3}$$

$$\lim_{0} \frac{(x - \sin x)^2}{(1 - \cos x)^3}$$

Oppgave 93 (1997–12–10: SIF5003 oppgave 8)
Bestem konvergensintervallet for potensrekken

$$\sum_{n=1}^{\infty} n x^n,$$

og finn et endelig uttrykk for summen i konvergensintervallet.

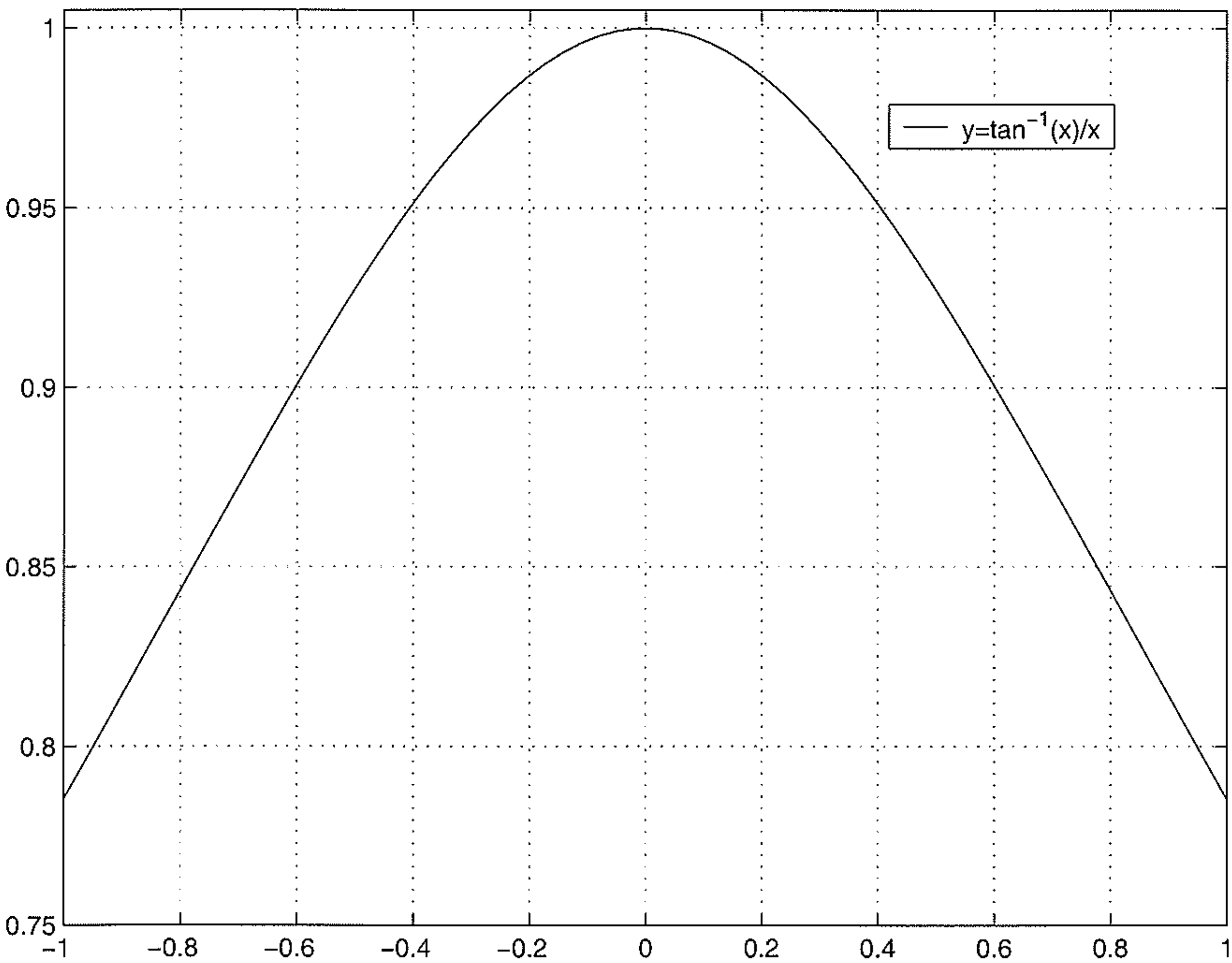
FINN TAYLORREKKA TIL $f(x) = \sin x$ OM PUNKTET $a = \pi/2$, OG BESTEM KONV.-RADIEN

$$\sum_{n=2}^{\infty} \frac{(-1)^n x^n}{n(n-1)}.$$

a) Finn konvergensområdet til denne rekken. La $f(x)$ være summen til rekken når x ligger i konvergensområdet og finn en rekkeutvikling for $f''(x)$.

b) Vis at $f(x) = (1+x)\ln(1+x) - x$.

c) Bruk a) og b) til å bestemme $\ln(3/2)$ med feil $< 1/250$.


y=tan^{-1}(x)/x
1
0.95
0.9
0.85
0.8
0.75
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1