18.3 Numerisk approksimasjon

De fleste ODL-er kan ikke løses analytisk!

Trenger numeriske metoder.

Problem:

$$\begin{cases} y' = f(x,y) , & x \in (a,b) \\ y(a) = y_0 , & x = a \end{cases}$$

Approksimasjon:

$$y(x+h) \overset{\text{Taylor}}{=} y(x) + y'(x)(x+h-x) + \tfrac{1}{2} y''(s)(x+h-x)^2$$

$$\overset{y'=f}{=} y(x) + h \cdot f(x, y(x)) + \tfrac{1}{2} h^2 y''(s)$$

$$(4) \qquad \simeq y(x) + h \cdot f(x, y(x)) \quad \text{når } h \text{ liten}$$

"tangentlinje"

La

$$h = \frac{b-a}{N} \quad \text{(steglengde)}$$

og

$$x_0 = a, \ \dots, \ x_n = a + nh, \ \dots, \ x_N = b$$

Da er:

$$y(x_0 + h) \overset{(4)}{\simeq} y_1 = y_0 + h \cdot f(x_0, y_0)$$

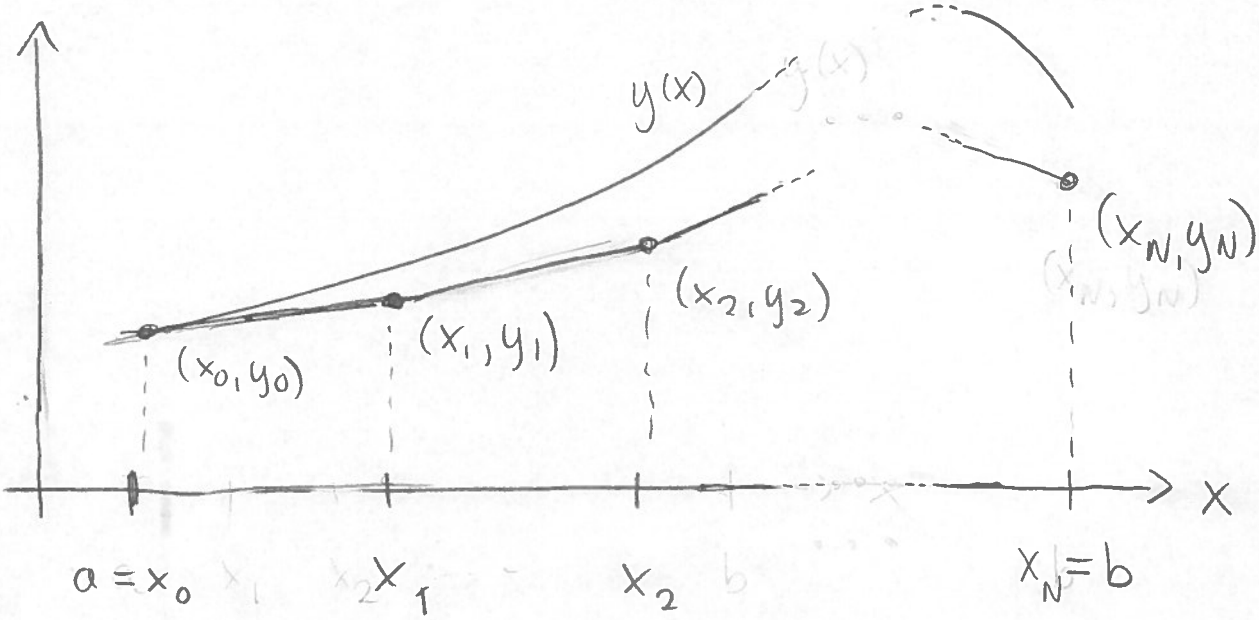
$$y(x_0 + 2h) \overset{(4)}{\simeq} y_2 = y_1 + h \cdot f(x_1, y_1)$$

$$\vdots$$

Eulers metode:

$$y_{n+1} = y_n + h\,f(x_n, y_n), \quad n = 0, 1, 2, \ldots$$

$$(y_0 = y(a))$$



Feil: $\max\limits_{n=0,\ldots,N} |y(x_n) - y_n| \leq K \cdot h \xrightarrow{h \to 0} 0$

$$\left[K = \frac{1}{2} \max_{a \leq x \leq b} |y''(x)|\, e^{L(b-a)} \text{ og } L \overset{f=f(y)}{=} \max_{y \in \mathbb{R}} |f'(y)|. \right]$$

Eks. 10: $y' = x + y, \quad x \in (0, 1); \quad y(0) = 0$

$h = \frac{1-0}{10}$: $x_0 = 0, \quad y_0 = 0$

$x_1 = \frac{1}{10}, \quad y_1 = y_0 + h(x_0 + y_0) = 0$

$x_2 = \frac{2}{10}, \quad y_2 = 0 + \frac{1}{10}\left(\frac{1}{10} + 0\right) = \frac{1}{100}$

$x_3 = \frac{3}{10}, \quad y_3 = \frac{1}{100} + \frac{1}{10}\left(\frac{2}{10} + \frac{1}{100}\right) = \frac{31}{1000}$

$\vdots$

8.)

$h = \frac{1}{10}$	$x_{10} = 1, \quad y_{10} \simeq 0{,}634$
$h = \frac{1}{20}$	$x_{20} = 1, \quad y_{20} \simeq 0{,}676$
$h = \frac{1}{50}$	$x_{50} = 1, \quad y_{50} \simeq 0{,}701$
Eksakt	$x = 1, \quad y(1) \simeq 0{,}718$

Mindre $h \Rightarrow$ bedre approks.

men

flere steg