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(continued after index)

Charles Chapman Pugh

Real Mathematical Analysis

With 133 Illustrations

Her har du en
omtale av Dedekindske
Snitt som kan være
nyttig.

Christian

Springer

An analogy is a shallow form of metaphor. It just asserts that two things are similar. Although simple, analogies can be a great help in accepting abstract concepts. When you travel from home to school, at first you are closer to home, and then you are closer to school. Somewhere there is a halfway stage in your journey. You *know* this, long before you study mathematics. So when a curve connects two points in a metric space (Chapter 2), you should expect that as a point "travels along the curve," somewhere it will be equidistant between the curve's endpoints. Reasoning by analogy is also referred to as "intuitive reasoning."

Moral Try to translate what you know of the real world to guess what is true in mathematics.

Two pieces of advice

A colleague of mine regularly gives his students an excellent piece of advice. When you confront a general problem and do not see how to solve it, make some extra hypotheses, and try to solve it then. If the problem is posed in n dimensions, try it first in two dimensions. If the problem assumes that some function is continuous, does it get easier for a differentiable function? The idea is to reduce an abstract problem to its simplest concrete manifestation, rather like a metaphor in reverse. At the minimum, look for at least one instance in which you can solve the problem, and build from there.

Moral If you do not see how to solve a problem in complete generality, first solve it in some special cases.

Here is the second piece of advice. Buy a notebook. In it keep a diary of your own opinions about the mathematics you are learning. Draw a picture to illustrate every definition, concept, and theorem.

2 Cuts

We begin at the beginning and discuss $\mathbb{R}$ = the system of all real numbers from a somewhat theological point of view. The current mathematics teaching trend treats the real number system $\mathbb{R}$ as a given — it is defined axiomatically. Ten or so of its properties are listed, called axioms of a complete ordered field, and the game becomes: deduce its other properties from the axioms. This is something of a fraud, considering that the entire structure of analysis is built on the real number system. For what if a system satisfying the axioms failed to exist? Then one would be studying the empty set! However, you need not take the existence of the real numbers on faith alone — we will give a concise mathematical proof of it.

It is reasonable to accept all grammar school arithmetic facts about

The set $\mathbb{N}$ of natural numbers, $1, 2, 3, 4, \ldots$.

The set $\mathbb{Z}$ of integers, $0, 1, -1, -2, 2, \ldots$.

The set $\mathbb{Q}$ of rational numbers p/q where p, q are integers, $q \neq 0$.

For example, we will admit without question facts like $2 + 2 = 4$, and laws like $a + b = b + a$ for rational numbers a, b. All facts you know about arithmetic involving integers or rational numbers are fair to use in homework exercises too.[†] It is clear that $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q}$. Now $\mathbb{Z}$ improves $\mathbb{N}$ because it contains negatives and $\mathbb{Q}$ improves $\mathbb{Z}$ because it contains reciprocals. $\mathbb{Z}$ legalizes subtraction and $\mathbb{Q}$ legalizes division. Still, $\mathbb{Q}$ needs further improvement. It doesn't admit irrational roots such as $\sqrt{2}$ or transcendental numbers such as π. We aim to go a step beyond $\mathbb{Q}$, completing it to form $\mathbb{R}$ so that

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}.$$

As an example of the fact that $\mathbb{Q}$ is incomplete we have

1 Theorem *No number r in $\mathbb{Q}$ has square equal to 2; i.e., $\sqrt{2} \notin \mathbb{Q}$.*

Proof To prove that every $r = p/q$ has $r^2 \neq 2$ we show that $p^2 \neq 2q^2$. It is fair to assume that p and q have no common factors since we would have canceled them out beforehand. Two integers without common factors can not both be even, so at least one of p, q is odd.

Case 1. p is odd. Then p^2 is odd while $2q^2$ is not. Therefore $p^2 \neq 2q^2$.

Case 2. p is even and q is odd. Then p^2 is divisible by 4 while $2q^2$ is not. Therefore $p^2 \neq 2q^2$. □

The set $\mathbb{Q}$ of rational numbers is incomplete. It has "gaps," one of which occurs at $\sqrt{2}$. These gaps are really more like pinholes; they have zero width. Incompleteness is what is *wrong* with $\mathbb{Q}$. Our goal is to complete $\mathbb{Q}$ by filling in its gaps. An elegant method to arrive at this goal is **Dedekind cuts** in which one visualizes real numbers as places at which a line may be cut with scissors. See Figure 3.

Definition A **cut** in $\mathbb{Q}$ is a pair of subsets A, B of $\mathbb{Q}$ such that

(a) $A \cup B = \mathbb{Q}$, $A \neq \emptyset$, $B \neq \emptyset$, $A \cap B = \emptyset$.

(b) If $a \in A$ and $b \in B$ then $a < b$.

(c) A contains no largest element.

[†] A subtler fact that you may find useful is the prime factorization theorem mentioned above. Any integer ≥ 2 can be factored into a product of prime numbers. For example, 120 is the product of primes $2 \cdot 2 \cdot 2 \cdot 3 \cdot 5$. Prime factorization is unique except for the order in which the factors appear. An easy consequence is that if a prime number p divides an integer k and if k is the product mn of integers then p divides m or it divides n. After all, by uniqueness, the prime factorization of k is just the product of the prime factorizations of m and n.

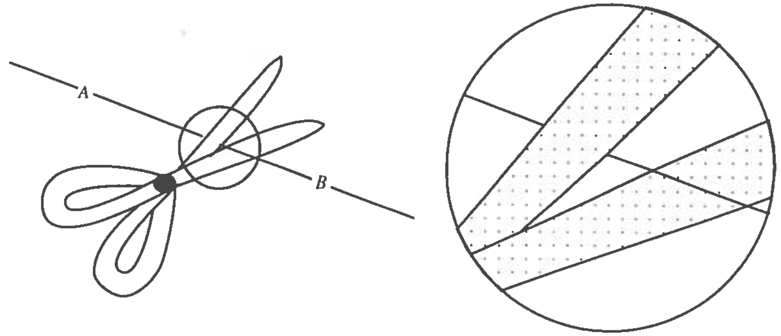

Figure 3 A Dedekind cut.

A is the left-hand part of the cut and B is the right-hand part. We denote the cut as $x = A|B$. Making a semantic leap, we now answer the question "what is a real number?"

Definition A **real number** is a cut in $\mathbb{Q}$.

$\mathbb{R}$ is the class† of all real numbers $x = A|B$. We will show that in a natural way $\mathbb{R}$ is a complete ordered field containing $\mathbb{Q}$. Before spelling out what this means, here are two examples of cuts.

(i) $A|B = \{r \in \mathbb{Q} : r < 1\} \mid \{r \in \mathbb{Q} : r \geq 1\}$.

(ii) $A|B = \{r \in \mathbb{Q} : r \leq 0 \text{ or } r^2 < 2\} \mid \{r \in \mathbb{Q} : r > 0 \text{ and } r^2 \geq 2\}$.

It is convenient to say that $A|B$ is a **rational cut** if it is like the cut in (i): for some fixed rational number c, A is the set of all rationals $< c$ while B is the rest of $\mathbb{Q}$. The B-set of a rational cut contains a smallest element c, and conversely, if $A|B$ is a cut in $\mathbb{Q}$ and B contains a smallest element c then $A|B$ is the rational cut at c. We write c^* for the rational cut at c. This lets us think of $\mathbb{Q} \subset \mathbb{R}$ by identifying c with c^*. It is like thinking of $\mathbb{Z}$ as a subset of $\mathbb{Q}$ since the integer n in $\mathbb{Z}$ can be thought of as the fraction $n/1$ in $\mathbb{Q}$. In the same way the rational number c in $\mathbb{Q}$ can be thought of as the cut at c. It is just a different way of looking at c. It is in this sense that we write

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}.$$

There is an order relation $x \leq y$ on cuts that fairly cries out for attention.

† The word "class" is used instead of the word "set" to emphasize that for now the members of $\mathbb{R}$ are set-pairs $A|B$, and not the numbers that belong to A or B. The notation $A|B$ could be shortened to A since B is just the rest of $\mathbb{Q}$. We write $A|B$, however, as a mnemonic device. It *looks* like a cut.

Definition The cut $x = A|B$ is **less than or equal** to the cut $y = C|D$ if $A \subset C$.

We write $x \leq y$ if x is less than or equal to y and we write $x < y$ if $x \leq y$ and $x \neq y$. If $x = A|B$ is less than $y = C|D$ then $A \subset C$ and $A \neq C$, so there is some $c_0 \in C \setminus A$. Since the A-set of a cut contains no largest element, there is also a $c_1 \in C$ with $c_0 < c_1$. All the rational numbers c with $c_0 \leq c \leq c_1$ belong to $C \setminus A$. Thus, $x < y$ implies that not only is $C \setminus A$ non-empty, but it contains infinitely many elements.

The property distinguishing $\mathbb{R}$ from $\mathbb{Q}$ and which is at the bottom of every significant theorem about $\mathbb{R}$ involves upper bounds and least upper bounds; or equivalently, lower bounds and greatest lower bounds.

$M \in \mathbb{R}$ is an **upper bound** for a set $S \subset \mathbb{R}$ if each $s \in S$ satisfies

$$s \leq M.$$

We also say that the set S is **bounded above** by M. An upper bound for S that is less than all other upper bounds for S is a **least upper bound** for S. The least upper bound for S is denoted l.u.b. (S). For example,

3 is an upper bound for the set of negative integers.

-1 is the least upper bound for the set of negative integers.

1 is the least upper bound for the set

$$\{x \in \mathbb{Q} : \exists n \in \mathbb{N} \text{ and } x = 1 - 1/n\}.$$

-100 is an upper bound for the empty set.

A least upper bound for S may or may not belong to S. This is why you should say "least upper bound *for* S" rather than "least upper bound *of* S."

2 Theorem *The set $\mathbb{R}$, constructed by means of Dedekind cuts, is* ***complete***† *in the sense that it satisfies the* ***Least Upper Bound Property:***

If S is a non-empty subset of $\mathbb{R}$ and is bounded above then in $\mathbb{R}$ there exists a least upper bound for S.

Proof Easy! Let $\mathcal{C} \subset \mathbb{R}$ be any non-empty collection of cuts which is bounded above, say by the cut $X|Y$. Define

$$C = \{a \in \mathbb{Q} : \text{ for some cut } A|B \in \mathcal{C}, a \in A\} \text{ and } D = \text{ the rest of } \mathbb{Q}.$$

It is easy to see that $z = C|D$ is a cut. Clearly, it is an upper bound for $\mathcal{C}$ since the "A" for every element of $\mathcal{C}$ is contained in C. Let $z' = C'|D'$

† There is another, related, sense in which $\mathbb{R}$ is complete. See Theorem 5 below.

be any upper bound for $\mathcal{C}$. By the assumption that $A|B \le C'|D'$ for all $A|B \in \mathcal{C}$, we see that the "A" for every member of $\mathcal{C}$ is contained in C'. Hence $C \subset C'$, so $z \le z'$. That is, among all upper bounds for $\mathcal{C}$, z is least. □

The simplicity of this proof is what makes cuts good. We go from $\mathbb{Q}$ to $\mathbb{R}$ by pure thought. To be more complete, as it were, we describe the natural arithmetic of cuts. Let cuts $x = A|B$ and $y = C|D$ be given. How do we add them? subtract them? ... Generally the answer is to do the corresponding operation to the elements comprising the two halves of the cuts, being careful about negative numbers. The sum of x and y is $x + y = E|F$ where

$$E = \{r \in \mathbb{Q} : \text{ for some } a \in A \text{ and } c \in C, r = a + c\}$$
$$F = \text{ the rest of } \mathbb{Q}.$$

It is easy to see that $E|F$ is a cut in $\mathbb{Q}$ and that it doesn't depend on the order in which x and y appear. That is, cut addition is well defined and $x + y = y + x$. The zero cut is 0^* and $0^* + x = x$ for all $x \in \mathbb{R}$. The additive inverse of $x = A|B$ is $-x = C|D$ where

$$C = \{r \in \mathbb{Q} : \text{ for some } b \in B, \text{ not the smallest element of } B, r = -b\}$$
$$D = \text{ the rest of } \mathbb{Q}.$$

Then $(-x) + x = 0^*$. Correspondingly, the difference of cuts is $x - y = x + (-y)$. Another property of cut addition is **associativity**:

$$(x + y) + z = x + (y + z).$$

This follows from the corresponding property of $\mathbb{Q}$.

Multiplication is trickier to define. It helps to first say that the cut $x = A|B$ is positive if $0^* < x$ or negative if $x < 0^*$. Since 0 lies in A or B, a cut is either positive, negative, or zero. If $x = A|B$ and $y = C|D$ are nonnegative cuts then their product is $x \cdot y = E|F$ where

$$E = \{r \in \mathbb{Q} \; : \; r < 0 \text{ or } \exists a \in A \text{ and } \exists c \in C \text{ such that } a > 0, c > 0, \text{ and } r = ac\},$$

and F is the rest of $\mathbb{Q}$. If x is positive and y is negative then we define the product to be $-(x \cdot (-y))$. Since x and $-y$ are both positive cuts this makes sense and is a negative cut. Similarly, if x is negative and y is positive then by definition their product is the negative cut $-((-x) \cdot y)$, while if x and y are both negative then their product is the positive cut $(-x) \cdot (-y)$.

Verifying the arithmetic properties for multiplication is tedious, to say the least, and somehow nothing seems to be gained by writing out every detail. (To pursue cut arithmetic further you could read Landau's classically boring book, *Foundations of Analysis*.) To get the flavor of it, let's check the commutativity of multiplication: $x \cdot y = y \cdot x$ for cuts $x = A|B$, $y = C|D$. If x, y are positive then

$$\{ac : a \in A, c \in C, a > 0, c > 0\} = \{ca : c \in C, a \in A, c > 0, a > 0\}$$

implies that $x \cdot y = y \cdot x$. If x is positive and y is negative then

$$x \cdot y = -(x \cdot (-y)) = -((-y) \cdot x) = y \cdot x.$$

The second equality holds because we have already checked commutativity for positive cuts. The remaining two cases are checked similarly. There are eight cases to check for associativity and eight more for distributivity. All are simple and we omit their proofs. The real point is that cut arithmetic can be defined and it satisfies the same field properties that $\mathbb{Q}$ does:

The operation of cut addition is
well defined, natural, commutative, associative, and
has inverses with respect to the neutral element 0^*.
The operation of cut multiplication
is well defined, natural, commutative, associative,
distributive over cut addition, and has inverses of
nonzero elements with respect to the neutral element 1^*.

By definition, a **field** is a system consisting of a set of elements and two operations, addition and multiplication, that have the preceding algebraic properties – commutativity, associativity, etc. Besides just existing, cut arithmetic is consistent with $\mathbb{Q}$ arithmetic in the sense that if $c, r \in \mathbb{Q}$ then $c^* + r^* = (c + r)^*$ and $c^* \cdot r^* = (cr)^*$. By definition, this is what we mean when we say that $\mathbb{Q}$ is a **subfield** of $\mathbb{R}$. The cut order enjoys the additional properties of

transitivity. $x < y < z$ implies $x < z$.

trichotomy. Either $x < y$, $y < x$, or $x = y$, but only one of the three things is true.

translation. $x < y$ implies $x + z < y + z$.

By definition, this is what we mean when we say that $\mathbb{R}$ is an **ordered field**. Besides, the product of positive cuts is positive and cut order is consistent with $\mathbb{Q}$ order: $c^* < r^*$ if and only if $c < r$ in $\mathbb{Q}$. By definition, this is what we mean when we say that $\mathbb{Q}$ is an ordered subfield of $\mathbb{R}$. To summarize

3 Theorem *The set $\mathbb{R}$ of all cuts in $\mathbb{Q}$ is a complete ordered field that contains $\mathbb{Q}$ as an ordered subfield.*

The **magnitude** or absolute value of $x \in \mathbb{R}$ is

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

Thus, $x \leq |x|$. A basic, constantly used fact about magnitude is the following.

4 Triangle Inequality *For all $x, y \in \mathbb{R}$, $|x + y| \leq |x| + |y|$.*

Proof The translation and transitivity properties of the order relation imply that adding y and $-y$ to the inequalities $x \leq |x|$ and $-x \leq |x|$ gives

$$\begin{aligned} x + y &\leq |x| + y \leq |x| + |y| \\ -x - y &\leq |x| - y \leq |x| + |y| . \end{aligned}$$

Since $x + y \leq |x| + |y|$ and $-(x + y) \leq |x| + |y|$, we infer that $|x + y| \leq |x| + |y|$ as asserted. □

Next, suppose we try the same cut construction in $\mathbb{R}$ that we did in $\mathbb{Q}$. Are there gaps in $\mathbb{R}$ that can be detected by cutting $\mathbb{R}$ with scissors? The natural definition of a cut in $\mathbb{R}$ is a division $\mathcal{A}|\mathcal{B}$ where $\mathcal{A}$ and $\mathcal{B}$ are disjoint, non-empty subcollections of $\mathbb{R}$ with $\mathcal{A} \cup \mathcal{B} = \mathbb{R}$, and $a < b$ for all $a \in \mathcal{A}$, $b \in \mathcal{B}$. Further, $\mathcal{A}$ contains no largest element. Now, each $b \in \mathcal{B}$ is an upper bound for $\mathcal{A}$. Therefore $y = \text{l. u. b.}(\mathcal{A})$ exists and $a \leq y \leq b$ for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$. By trichotomy,

$$\mathcal{A}|\mathcal{B} = \{x \in \mathbb{R} : x < y\} \mid \{x \in \mathbb{R} : x \geq y\}.$$

In other words, $\mathbb{R}$ has no gaps. Every cut in $\mathbb{R}$ occurs exactly *at* a real number.

Allied to the existence of $\mathbb{R}$ is its uniqueness. Any complete ordered field $\mathbb{F}$ containing $\mathbb{Q}$ as an ordered subfield corresponds to $\mathbb{R}$ in a way preserving all the ordered field structure. To see this, take any $\varphi \in \mathbb{F}$ and associate to it the cut $A|B$ where

$$A = \{r \in \mathbb{Q} : r < \varphi \text{ in } \mathbb{F}\}.$$

This correspondence makes $\mathbb{F}$ equivalent to $\mathbb{R}$.

Upshot The real number system $\mathbb{R}$ exists and it satisfies the properties of a complete ordered field; the properties are not assumed as axioms, but are proved by logically analyzing the Dedekind construction of $\mathbb{R}$. Having gone through all this cut rigmarole, it must be remarked that it is a rare working mathematician who actually thinks of $\mathbb{R}$ as a complete ordered field or as the set of all cuts in $\mathbb{Q}$. Rather, he or she thinks of $\mathbb{R}$ as points on the x-axis, just as in calculus. You too should picture $\mathbb{R}$ this way, the only benefit of the cut derivation being that you should now unhesitatingly accept the least upper bound property of $\mathbb{R}$ as a true fact.

Note $\pm\infty$ are not real numbers since $\mathbb{Q}|\emptyset$ and $\emptyset|\mathbb{Q}$ are not cuts. Although some mathematicians think of $\mathbb{R}$ together with $-\infty$ and $+\infty$ as an "extended real number system," it is simpler to leave well enough alone and just deal with $\mathbb{R}$ itself. Nevertheless, it is convenient to write expressions like "$x \to \infty$" to indicate that a real variable x grows larger and larger without bound.

If S is a non-empty subset of $\mathbb{R}$ then its **supremum** is its least upper bound when S is bounded above and is said to be $+\infty$ otherwise; its **infimum** is its greatest lower bound when S is bounded below and is said to be $-\infty$ otherwise. (In Exercise 17 you are asked to invent the notion of greatest lower bound.) By definition the supremum of the empty set is $-\infty$. This is reasonable, considering that every real number, no matter how negative, is an upper bound for $\emptyset$, and the least upper bound should be as far leftward as possible, namely $-\infty$. Similarly, the infimum of the empty set is $+\infty$. We write $\sup S$ and $\inf S$ for the supremum and infimum of S.

Cauchy sequences

As mentioned above there is a second sense in which $\mathbb{R}$ is complete. It involves the concept of convergent sequences. Let $a_1, a_2, a_3, a_4, \dots = (a_n)$, $n \in \mathbb{N}$, be a sequence of real numbers. The sequence (a_n) **converges to the limit** $b \in \mathbb{R}$ as $n \to \infty$ provided that for each $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$|a_n - b| < \epsilon.$$

The statistician's language is evocative here. Think of $n = 1, 2, \dots$ as a sequence of times and say that the sequence (a_n) converges to b provided that *eventually* all its terms nearly equal b. In symbols,

$$\forall \epsilon > 0 \ \exists N \in \mathbb{N} \text{ such that } n \geq N \Rightarrow |a_n - b| < \epsilon.$$

If the limit b exists it is not hard to see that it is unique, and we write

$$\lim_{n\to\infty} a_n = b \text{ or } a_n \to b.$$

Suppose that $\lim_{n\to\infty} a_n = b$. Since all the numbers a_n are eventually near b they are all near each other; i.e., every convergent sequence obeys a **Cauchy condition**:

$$\forall \epsilon > 0 \ \exists N \in \mathbb{N} \text{ such that } n, m \geq N \Rightarrow |a_n - a_m| < \epsilon.$$

The converse of this fact is a fundamental property of $\mathbb{R}$.

5 Theorem *$\mathbb{R}$ is **complete** with respect to Cauchy sequences in the sense that if (a_n) is a sequence of real numbers obeying a Cauchy condition then it converges to a limit in $\mathbb{R}$.*

Proof Let A be the set of real numbers comprising the sequence (a_n),

$$A = \{x \in \mathbb{R} : \exists n \in \mathbb{N} \text{ and } a_n = x\}.$$

We first observe that A is a bounded set in $\mathbb{R}$. Taking $\epsilon = 1$ in the Cauchy condition, there is an integer N_1 such that for all $n, m \geq N_1$, $|a_n - a_m| < 1$. Then, for each $n \geq N_1$

$$(10) \qquad |a_n - a_{N_1}| < 1.$$

Clearly the finite set $a_1, a_2, \ldots, a_{N_1}, a_{N_1} - 1, a_{N_1} + 1$ is bounded, (any finite set is bounded); say all its elements belong to the interval $[-M, M]$. According to (10), $[-M, M]$ contains A so A is bounded.

Next, consider the set

$$S = \{s \in [-M, M] : \exists \text{ infinitely many } n \in \mathbb{N}, \text{ for which } a_n \geq s\}.$$

That is, $a_n \geq s$ infinitely often. Clearly $-M \in S$ and S is bounded above by M. According to the least upper bound property of $\mathbb{R}$ there exists $b \in \mathbb{R}$, $b = \text{l. u. b. } S$. We claim that the sequence (a_n) converges to b.

Given $\epsilon > 0$ we must show that there exists an N such that for all $n \geq N$, $|a_n - b| < \epsilon$. The Cauchy condition provides an N_2 such that

$$(11) \qquad m, n \geq N_2 \quad \Rightarrow \quad |a_m - a_n| < \frac{\epsilon}{2}.$$

All elements of S are $\leq b$, so the larger number $b + \epsilon/2$ does not belong to S. Only finitely often does a_n exceed $b + \epsilon/2$. That is, for some $N_3 \geq N_2$,

$$n \geq N_3 \quad \Rightarrow \quad a_n \leq b + \frac{\epsilon}{2}.$$

Since b is a least upper bound for S, the smaller number $b - \epsilon/2$ can not also be an upper bound for S. Some $s \in S$ is $> b - \epsilon/2$, which implies that $a_n \geq s > b - \epsilon/2$ infinitely often. In particular, there exists $N \geq N_3$ such that $a_N > b - \epsilon/2$. Since $N \geq N_3$, we have $a_N \leq b + \epsilon/2$ and so

$$a_N \in (b - \epsilon/2, \ b + \epsilon/2].$$

Since $N \geq N_2$, (11) implies

$$|a_n - b| \leq |a_n - a_N| + |a_N - b| < \epsilon,$$

which verifies convergence. □

Restating Theorem 5 gives the

6 Cauchy Convergence Criterion for sequences *A sequence (a_n) in $\mathbb{R}$ converges if and only if*

$$\forall \epsilon > 0 \ \exists N \in \mathbb{N} \text{ such that } n, m \geq N \Rightarrow |a_n - a_m| < \epsilon.$$

Further description of $\mathbb{R}$

The elements of $\mathbb{R} \setminus \mathbb{Q}$ are **irrational numbers**. If x is irrational and r is rational then $y = x + r$ is irrational. For if y is rational then so is $y - r = x$, the difference of rationals being rational. Similarly, if $r \neq 0$ then rx is irrational. It follows that the reciprocal of an irrational number is irrational. From these observations we will show that the rational and irrational numbers are thoroughly mixed up with each other.

Let $a < b$ be given in $\mathbb{R}$. Define the **intervals** (a, b) and $[a, b]$ as

$$(a, b) = \{x \in \mathbb{R} : a < x < b\}$$
$$[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}.$$

7 Theorem *Every interval (a, b), no matter how small, contains both rational and irrational numbers.*

Proof This is certainly true of the interval $(0, 1)$ since it contains the numbers $1/2$ and $1/\sqrt{2}$. For the general interval (a, b), think of a, b as cuts $a = A|A'$, $b = B|B'$. The fact that $a < b$ implies the set $B \setminus A$ contains two distinct rational numbers, say r, s. Thus $a \leq r < s \leq b$. The transformation

$$T : t \mapsto r + (s - r)t$$

sends the interval $(0, 1)$ to the interval (r, s). Since r, s, and $s-r$ are rational, T sends rationals to rationals and irrationals to irrationals. That is, (r, s) contains both rationals and irrationals, and so does the larger interval (a, b). □

Theorem 7 expresses the fact that between any two rational numbers lies an irrational number; and between any two irrational numbers lies a rational number. This is a fact worth thinking about for it seems implausible at first. Spend some time trying to picture the situation, especially in light of the following related facts:

(a) There is no first (i.e., smallest) rational number in the interval $(0, 1)$.
(b) There is no first irrational number in the interval $(0, 1)$.
(c) There are strictly more irrational numbers in the interval $(0, 1)$ (in the cardinality sense explained in Section 4) than there are rational numbers.

The transformation in the proof of Theorem 7 shows that the real line is like rubber: stretch it out and it never breaks.

A somewhat obscure and trivial fact about $\mathbb{R}$ is its **Archimedean property**: for each $x \in \mathbb{R}$ there is an integer n that is greater than x. In other words, there exist arbitrarily large integers. The Archimedean property is true for $\mathbb{Q}$ since $p/q \leq |p|$. It follows that it is true for $\mathbb{R}$. Given $x = A|B$, just choose a rational number $r \in B$ and an integer $n > r$. Then $n > x$. An equivalent way to state the Archimedean property is that there exist arbitrarily small reciprocals of integers.

Mildly interesting is the existence of ordered fields for which the Archimedean property fails. One example is the field $\mathbb{R}(x)$ of rational functions with real coefficients. Each such function is of the form

$$R(x) = \frac{p(x)}{q(x)}$$

where p and q are polynomials with real coefficients and q is not the zero polynomial. (It does not matter that $q(x) = 0$ at a finite number of points.) Addition and multiplication are defined in the usual fashion of high school algebra, and it is easy to see that $\mathbb{R}(x)$ is a field. The order relation on $\mathbb{R}(x)$ is also easy to define. If $R(x) > 0$ for all sufficiently large x then we say that R is positive in $\mathbb{R}(x)$, and if $R - S$ is positive then we write $S < R$. Since a nonzero rational function vanishes (has value zero) at only finitely many $x \in \mathbb{R}$, we get trichotomy: either $R = S$, $R < S$, or $S < R$. (To be rigorous, we need to prove that the values of a rational function do not change sign for x large enough.) The other order properties are equally easy to check, and $\mathbb{R}(x)$ is an ordered field.

Is $\mathbb{R}(x)$ Archimedean? That is, given $R \in \mathbb{R}(x)$, does there exist a natural number $n \in \mathbb{R}(x)$ such that $R < n$? (A number n is the rational function whose numerator is the constant polynomial $p(x) = n$, a polynomial of degree zero, and whose denominator is the constant polynomial $q(x) = 1$.) The answer is "no." Take $R(x) = x/1$. The numerator is x and the denominator is 1. Clearly, we have $n < x$, not the opposite, so $\mathbb{R}(x)$ fails to be Archimedean.

The same remarks hold for any positive rational function $R = p(x)/q(x)$ where the degree of p exceeds the degree of q. In $\mathbb{R}(x)$, R is never less than a natural number. (You might ask yourself: exactly which rational functions are less than n?)

The ϵ-principle

Finally let us note a nearly trivial principle that turns out to be invaluable in deriving inequalities and equalities in $\mathbb{R}$.

8 Theorem (ϵ-principle) *If a, b are real numbers and if for each $\epsilon > 0$, $a \leq b + \epsilon$, then $a \leq b$. If x, y are real numbers and for each $\epsilon > 0$, $|x - y| \leq \epsilon$, then $x = y$.*

Proof Trichotomy implies that either $a \leq b$ or $a > b$. In the latter case we can choose ϵ, $0 < \epsilon < a - b$ and get the absurdity

$$\epsilon < a - b \leq \epsilon.$$

Hence $a \leq b$. Similarly, if $x \neq y$ then choosing ϵ, $0 < \epsilon < |x - y|$ gives the contradiction $\epsilon < |x - y| \leq \epsilon$. Hence $x = y$. See also Exercise 11. □

3 Euclidean Space

Given sets A and B, the **Cartesian product** of A and B is the set $A \times B$ of all ordered pairs (a, b) such that $a \in A$ and $b \in B$. (The name comes from Descartes who pioneered the idea of the (x, y)-coordinate system in geometry.) See Figure 4.

The Cartesian product of $\mathbb{R}$ with itself m times is denoted $\mathbb{R}^m$. Elements of $\mathbb{R}^m$ are vectors, ordered m-tuples of real numbers, $(x_1, \ldots, x_m)$. In this terminology, real numbers are called scalars and $\mathbb{R}$ is called the scalar field. When vectors are added, subtracted, and multiplied by scalars according to the rules