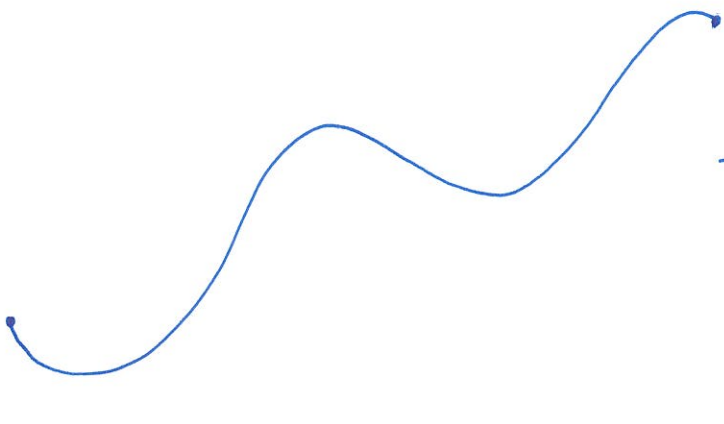
$$L = \int_0^{2\pi} \sqrt{2}\, e^t \, dt = \sqrt{2}\left[e^t\right]_0^{2\pi} = \underline{\underline{\sqrt{2}\left(e^{2\pi}-1\right)}}$$

Krumning og smygsirkel

La $\mathbb{T}(s)$ være enhetstangentvektoren til en glatt kurve C ($\mathbb{T}(s) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$).

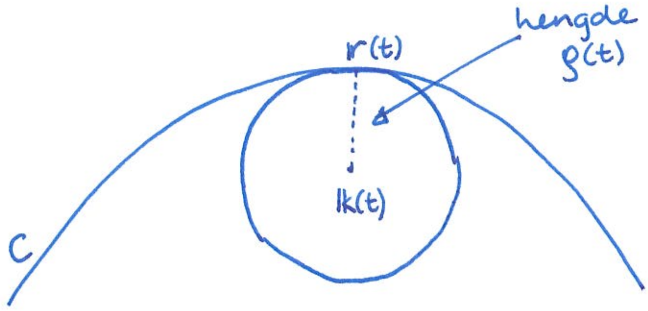


Hvor krapp er svingene?

Definisjon: Krumningen til den glatte kurven $\mathbf{r} = \mathbf{r}(s)$ i punktet $\mathbf{r}(s)$ er definert ved

$$\kappa = \left|\frac{d\mathbb{T}}{ds}\right| = \left|\frac{d\mathbb{T}}{dt}\cdot\frac{dt}{ds}\right| = \left|\frac{d\mathbb{T}}{ds}\cdot\frac{1}{|\mathbf{r}'(t)|}\right|.$$

Smygsirkelen



C : glatt kurve

$\mathbb{r}(t)$: tangeringspunkt

$\mathbb{k}(t)$: krumningssenter

$\rho(t)$: radien

For å finne smygsirkelen trenger vi radien ρ og senteret $\mathbb{k}$.

Vi finner da først $\kappa(t)$ og enhetsnormalen $\mathbb{N}(t)$.

1) $\rho(t)$: $\rho(t) = \dfrac{1}{\kappa(t)}$, der $\kappa(t) = \left| \dfrac{d\mathbb{T}}{ds} \right|$.

2) $\mathbb{k}(t)$: $\mathbb{k}(t) = \mathbb{r}(t) + \rho\,\mathbb{N}(t)$, der

$\mathbb{r}(t)$ er den parametriserte kurven, og $\mathbb{N}(t) = \dfrac{\mathbb{a}(t) - a_{\mathbb{T}}(t)\cdot\mathbb{T}(t)}{|\mathbb{a}(t) - a_{\mathbb{T}}(t)\cdot\mathbb{T}(t)|}$

$\left(a_{\mathbb{T}}(t) = \mathbb{a}(t)\cdot\mathbb{T}(t)\ ,\ \mathbb{T}(t) = \dfrac{\mathbb{v}(t)}{|\mathbb{v}(t)|} \right)$

Ekstra verktøy:

Dekomponering av $\mathbb{a}(t)$ hjelper oss å finne smygsirkelen:

$$\mathbb{a} = \underbrace{a_{\mathbb{T}}\mathbb{T}}_{\text{tang.}} + \underbrace{a_{\mathbb{N}}\mathbb{N}}_{\text{norm.}}$$

$$a_{\mathbb{T}} = \mathbb{a}\cdot\mathbb{T} \quad \text{og} \quad a_{\mathbb{N}} = |\mathbb{a} - a_{\mathbb{T}}\mathbb{T}| .$$

Eksempel 6

Finn akselerasjonsvektor til $\mathbb{r}(t) = [t^2, t^{3/2}]$ i punktet $t = 1$.

Dekomponer i tangensialkomponent og normalkomponent.

Finn krumning og smygsirkel.

$$\mathbb{r}(t) = [t^2, t^{3/2}]$$

$$\mathbb{v}(t) = \mathbb{r}'(t) = [2t, \tfrac{3}{2}t^{1/2}]$$

$$\mathbb{a}(t) = \mathbb{v}'(t) = [2, \tfrac{3}{4}t^{-1/2}]$$

Dekomponering:

$$\mathbb{a} = a_{\mathbb{T}} \mathbb{T} + a_{\mathbb{N}} \mathbb{N} . \qquad (*)$$

$$a_{\mathbb{T}} = \mathbb{a} \cdot \mathbb{T} = \mathbb{a} \cdot \frac{\mathbb{v}(t)}{|\mathbb{v}(t)|}$$

$$= \left[2, \tfrac{3}{4} t^{-1/2}\right] \cdot \left[2t, \tfrac{3}{2} t^{1/2}\right] \frac{1}{\sqrt{4t^2 + \frac{9}{4} t}}$$

Siden vi kun trenger for $t = 1$,
setter vi inn før vi regner videre:

$$a_{\mathbb{T}}(1) = \left[2, \tfrac{3}{4}\right] \cdot \left[2, \tfrac{3}{2}\right] \cdot \frac{1}{\frac{5}{2}}$$

$$= \left(4 + \frac{9}{8}\right) \cdot \frac{2}{5} = \frac{41}{20} .$$

$$\mathbb{T} = \frac{\mathbb{v}(t)}{|\mathbb{v}(t)|} = \frac{\left[2t, \frac{3}{2} t^{1/2}\right]}{\sqrt{4t^2 + \frac{9}{4} t}}$$

$$\mathbb{T}(1) = \frac{1}{\frac{5}{2}} \left[2, \tfrac{3}{2}\right] = \frac{2}{5} \left[2, \tfrac{3}{2}\right] = \frac{1}{5} [4, 3] .$$

Tangensialkomp: $a_{\mathbb{T}} \mathbb{T} = \frac{41}{20} \cdot \frac{1}{5} [4, 3]$

$$= \frac{41}{100} [4, 3] .$$

P.g.a. (*), må

$$a_{\mathbb{N}} \mathbb{N} = \mathbb{a} - a_{\mathbb{T}} \mathbb{T}$$

$$= [2, \tfrac{3}{4}] - \frac{41}{100}[4, 3]$$

$$= [\tfrac{9}{25}, -\tfrac{12}{25}] = \frac{3}{25}[3, -4],$$

som altså er normalkomponenten.

Da gjenstår krumningen og smygsirkelen:

(s. 412 LHL)

Siden $a_{\mathbb{N}} = |\mathbb{a} - a_{\mathbb{T}}\mathbb{T}| = \kappa v^2$, får vi

$$a_{\mathbb{N}}(1) = |[\tfrac{9}{25}, -\tfrac{12}{25}]| = \frac{3}{5} = \kappa(1) \cdot \frac{25}{4}$$

$$\underline{\kappa(1) = \frac{12}{125}}.$$

$$\rho(1) = \frac{1}{\kappa(1)} = \frac{125}{12},$$ som er senteret i smygsirkelen.

$$\mathbb{N}(1) = \frac{\mathbb{a}(1) - a_{\mathbb{T}} \cdot \mathbb{T}}{a_{\mathbb{N}}} = \frac{\left[2, \frac{3}{4}\right] - \frac{41}{100}\left[4, 3\right]}{\frac{3}{5}}$$

$$= \frac{5}{3}\left[\frac{9}{25}, -\frac{12}{25}\right] = \frac{15}{75}\left[3, -4\right] .$$

$$= \frac{1}{5}\left[3, -4\right]$$

$$\mathbb{k}(1) = \mathbb{r}(1) + \rho(1)\,\mathbb{N}(1)$$

$$= [1, 1] + \frac{125}{12} \cdot \frac{1}{5} \quad [\cdot 3, -4]$$

$$= \left[\frac{29}{4}, -\frac{22}{3}\right] .$$