DEFINITION Line Integral of a Function along a Curve S. 1020

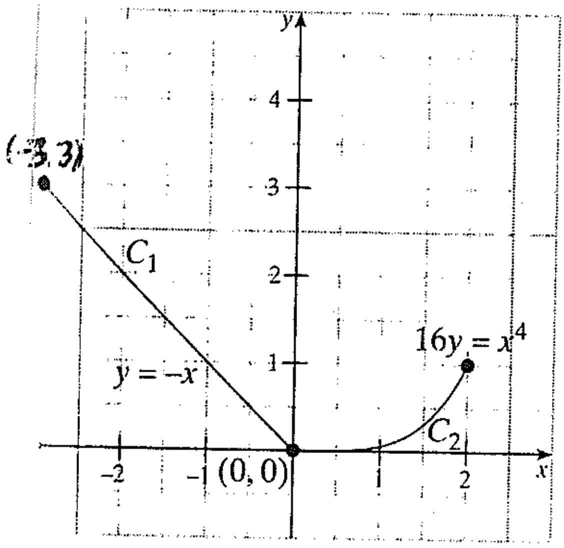
Suppose that the function $f(x, y, z)$ is defined at each point of the smooth curve C parametrized as in (1). Then the **line integral of f along C** is defined by

$$\int_C f(x, y, z)\, ds = \lim_{\Delta t \to 0} \sum_{i=1}^{n} f(x(t_i^\star), y(t_i^\star), z(t_i^\star))\, \Delta s_i,$$

provided that this limit exists.

Evaluate the line integral $\int_C x^2 z\, ds$, where C is the helix $x = \cos t$, $y = 2t$, $z = \sin t$ for $0 \le t \le \pi$.

Evaluate the line integral $\int_C xy\, ds$, where C consists of the line segment C_1 from $(-3, 3)$ to $(0, 0)$, followed by the portion of the curve C_2: $16y = x^4$ between $(0, 0)$ and $(2, 1)$.



Evaluate the line integral

$$\int_C [y\, dx - z\, dy + x\, dz]$$

where C is the curve with parametric equations $x = t^2$, $y = e^{-t}$, $z = e^t$ for $0 \le t \le 1$

PROPERTIES

$$\int_C k f \, ds = k \int_C f \, ds \quad , k \in \mathbb{R}$$

$$\int_C (f+g) \, ds = \int f \, ds + \int g \, ds$$

$$\int_{-C} f \, ds = - \int_C f \, ds$$

$$\int_C f ds = \int_{C_1} f ds + \int_{C_2} f ds + \dots + \int_{C_n} f \, ds \quad , \quad C = C_1 \cup C_2 \cup \dots \cup C_n$$

WIRE:

$$dm = \delta(x,y,z) \, ds$$

s. 1022

$$m = \int_C dm = \int_C \delta(x,y,z) \, ds$$

$$\bar{x} = \frac{1}{m} \int_C x \, dm$$

$$\bar{y} = \frac{1}{m} \int_C y \, dm$$

$$\bar{z} = \frac{1}{m} \int_C z \, dm$$

$$I_z = \int_C r^2 dm$$

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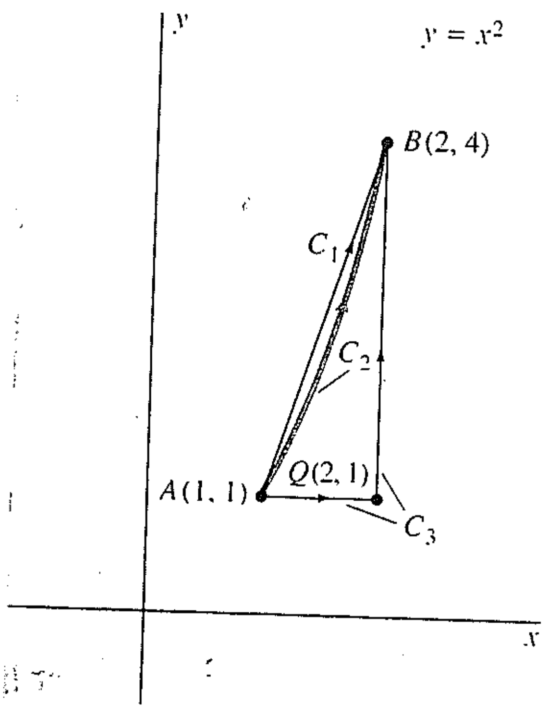
EXAMPLE 5 Evaluate the line integral S. 1025

$$\int_C y\,dx + 2x\,dy$$

for each of these three curves C (Fig. 15.2.9):

C_1 The straight line segment in the plane from $A(1, 1)$ to $B(2, 4)$;
C_2 The plane path from $A(1, 1)$ to $B(2, 4)$ along the graph of the parabola $y = x^2$ and
C_3 The straight line in the plane from $A(1, 1)$ to $Q(2, 1)$ followed by the straight line from $Q(2, 1)$ to $B(2, 4)$.



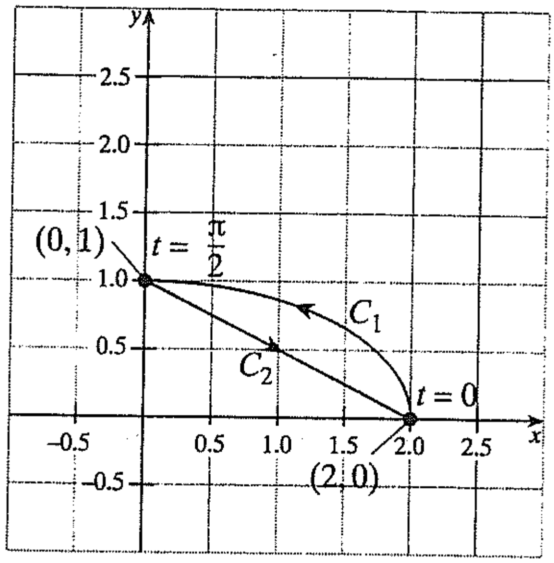
S. 1027

THEOREM 1 Equivalent Line Integrals

Suppose that the vector field $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ has continuous component functions and that $\mathbf{T}$ is the unit tangent vector to the smooth curve C. Then

$$\int_C \mathbf{F}\cdot\mathbf{T}\,ds = \int_C P\,dx + Q\,dy + R\,dz.$$

An object moves in the force field $\mathbf{F} = y^2\mathbf{i} + 2(x+1)y\mathbf{j}$. How much work is performed as the object moves from the point $(2, 0)$ counterclockwise along the elliptical path $x^2 + 4y^2 = 4$ to $(0, 1)$, then back to $(2, 0)$ along the line segment joining the two points, as shown in Figure 13.13.



S. 1030

THEOREM 1 The Fundamental Theorem for Line Integrals

Let f be a function of two or three variables and let C be a smooth curve parametrized by the vector-valued function $\mathbf{r}(t)$ for $a \leqq t \leqq b$. If f is continuously differentiable at each point of C, then

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$

Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{R}$, where

$$\mathbf{F} = \nabla(e^x \sin y - xy - 2y)$$

and C is the path described by $\mathbf{R}(t) = \left[t^3 \sin \frac{\pi}{2}t\right]\mathbf{i} - \left[\frac{\pi}{2}\cos\left(\frac{\pi}{2}t + \frac{\pi}{2}\right)\right]\mathbf{j}$ for $0 \leq t \leq 1$