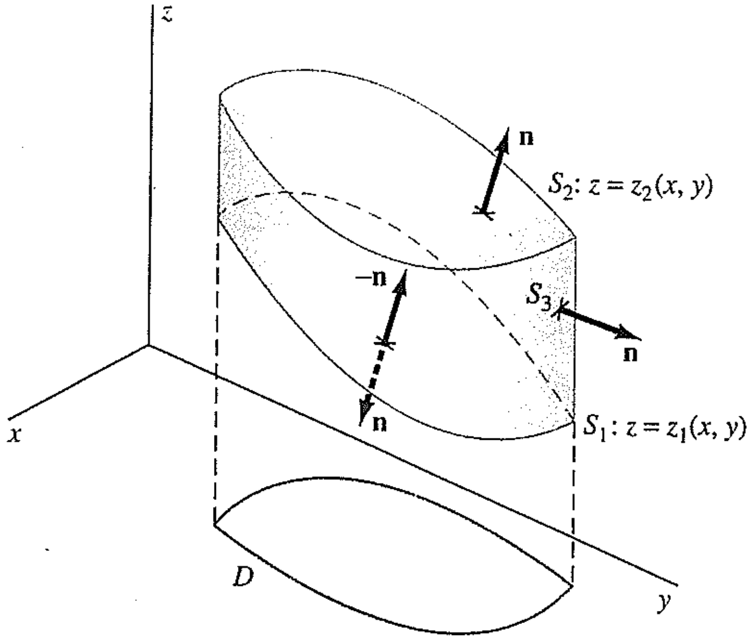
THE DIVERGENCE THEOREM

S. 1058

Suppose that S is a closed piecewise smooth surface that bounds the space region T and let the *outer* unit normal vector field $\mathbf{n}$ be continuous on each smooth piece of S. If the vector field $\mathbf{F}$ is continuously differentiable on T, then

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_T \nabla \cdot \mathbf{F}\, dV.$$

$$\iint_S P\, dy\, dz + Q\, dz\, dx + R\, dx\, dy = \iiint_T \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dV. \tag{4}$$

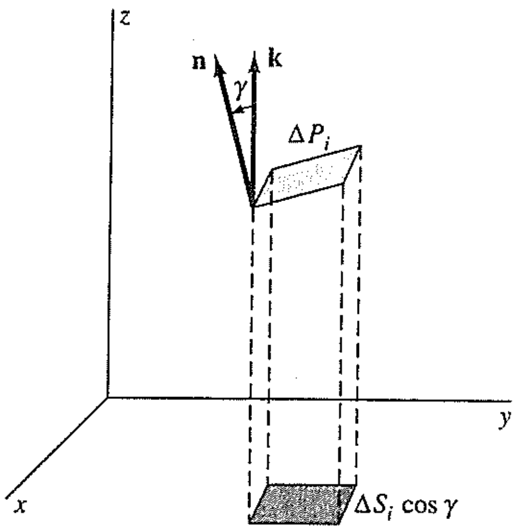


S. 1051

$$\iint_S P\, dy\, dz + Q\, dz\, dx + R\, dx\, dy = \iint_S (P \cos\alpha + Q \cos\beta + R \cos\gamma)\, dS; \tag{16}$$

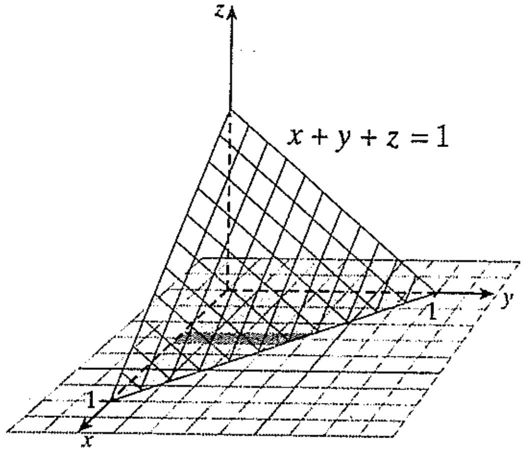
S. 1049

$$\iint_S f(x, y, z)\, dS = \iint_D f(x, y, h(x, y)) \sqrt{1 + \left(\frac{\partial h}{\partial x}\right)^2 + \left(\frac{\partial h}{\partial y}\right)^2}\, dx\, dy. \tag{9}$$

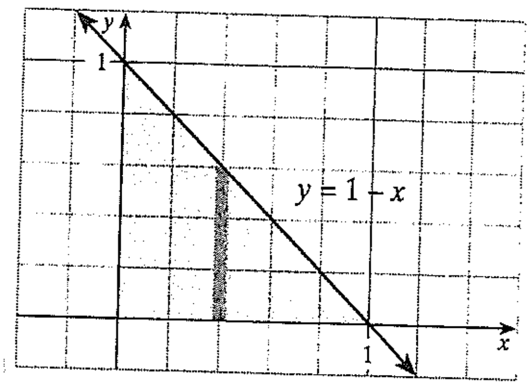


Evaluating a surface integral using the divergence theorem

Evaluate $\iint_S \mathbf{F}\cdot\hat{n}\, dS$, where $\mathbf{F} = x^2\mathbf{i} + xy\mathbf{j} + x^3y^3\mathbf{k}$ and S is the surface of the tetrahedron bounded by the plane $x + y + z = 1$ and the coordinate planes, with outward unit normal vector $\hat{n}$ (see Figure 13.47).



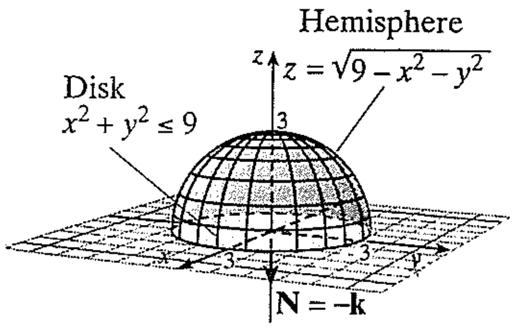
a. The surface S



b. Projected region in the xy-plane

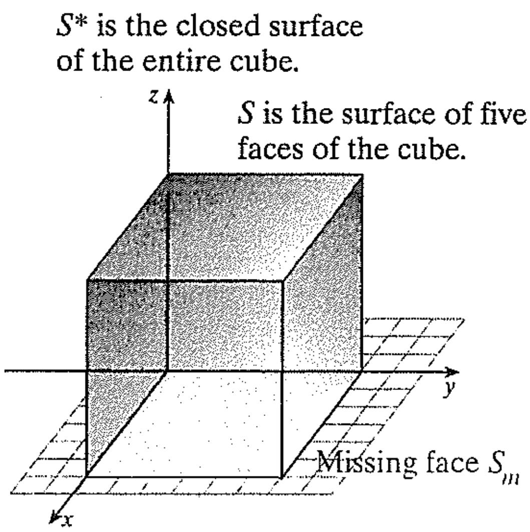
Verifying the divergence theorem for a particular solid

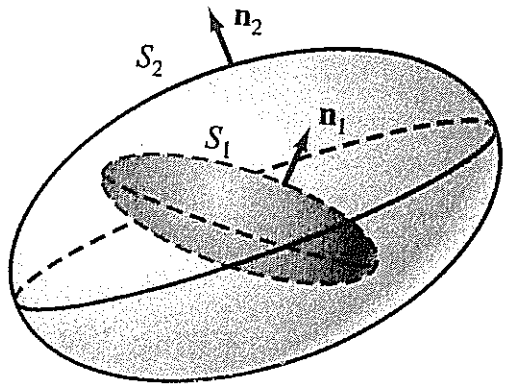
Let $\mathbf{F} = 2x\mathbf{i} - 3y\mathbf{j} + 5z\mathbf{k}$, and let S be the hemisphere $z = \sqrt{9 - x^2 - y^2}$ together witl the disk $x^2 + y^2 \leq 9$ in the xy-plane. Verify the divergence theorem.



Evaluating a surface integral over an open surface

Evaluate $\iint_S \mathbf{F}\cdot\hat{n}\, dS$, where $\mathbf{F} = xy\mathbf{i} - z^2\mathbf{k}$ and S is the surface of the upper 5 faces of the unit cube $0 \leq x \leq 1, 0 \leq y \leq 1, 0 < z \leq 1$, as shown in Figure below.



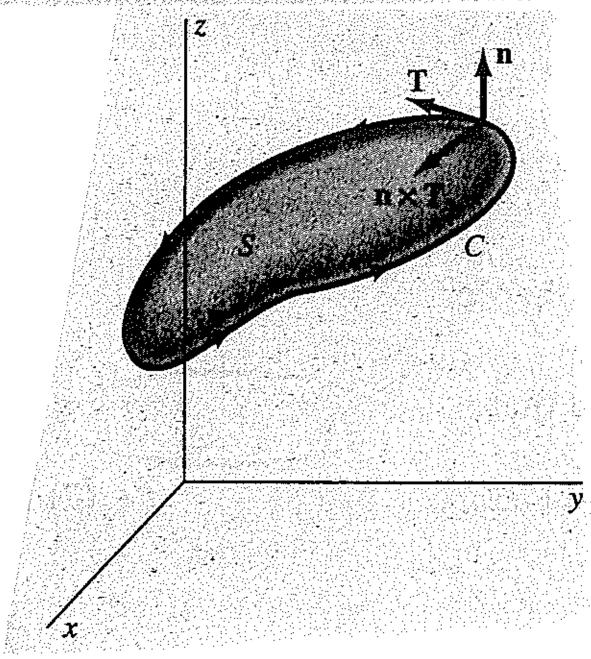


STOKES' THEOREM

s.1065

Let S be an oriented, bounded, and piecewise smooth surface in space with positively oriented boundary C and unit normal vector field $\mathbf{n}$. Suppose that $\mathbf{T}$ is a positively oriented unit vector field tangent to C. If the vector field $\mathbf{F}$ is continuously differentiable in a space region containing S, then

$$\oint_C \mathbf{F}\cdot\mathbf{T}\,ds = \iint_S (\operatorname{curl}\mathbf{F})\cdot\mathbf{n}\,dS.$$



s.1066

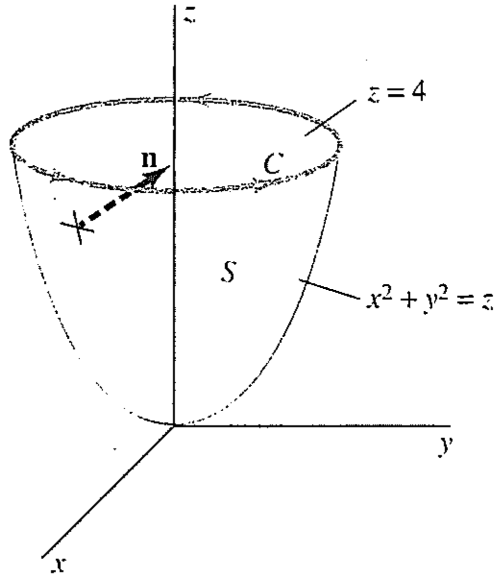
$$\oint_C P\,dx + Q\,dy + R\,dz$$
$$= \iint_S \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) dy\,dz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) dz\,dx + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy.$$

EXAMPLE 2 Apply Stokes' theorem to evaluate

s. 1067

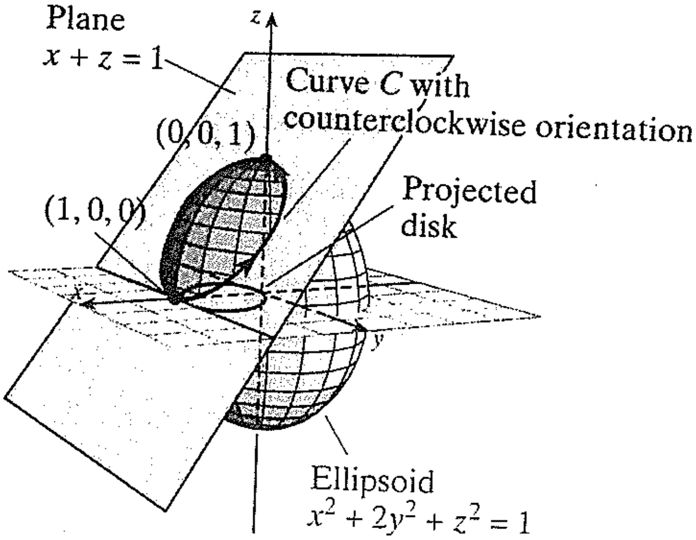
$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS,$$

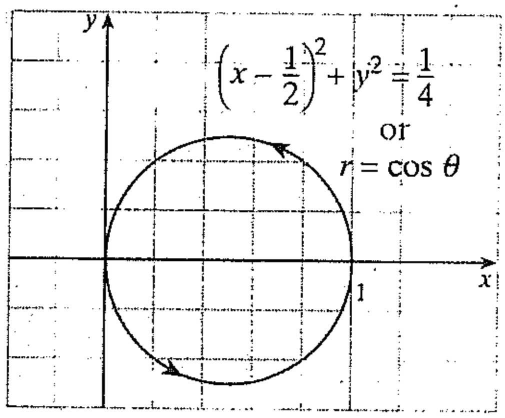
where $\mathbf{F} = 3z\,\mathbf{i} + 5x\mathbf{j} - 2y\mathbf{k}$ and S is the part of the parabolic surface $z = x^2 + y^2$ that lies below the plane $z = 4$ and whose orientation is given by the upper unit normal vector (Fig. 15.7.4).



Using Stokes' theorem to evaluate a line integral

Evaluate $\oint_C (\frac{1}{2}y^2\, dx + z\, dy + x\, dz)$, where C is the curve of intersection of the plane $x + z = 1$ and the ellipsoid $x^2 + 2y^2 + z^2 = 1$, oriented counterclockwise as viewed from above (see Figure 13.41).





THEOREM 1 Conservative and Irrotational Fields s. 1068

Let $\mathbf{F}$ be a vector field with continuous first-order partial derivatives in a simply connected region D in space. Then the vector field $\mathbf{F}$ is irrotational if and only if it is conservative; that is, $\nabla \times \mathbf{F} \equiv \mathbf{0}$ in D if and only if $\mathbf{F} = \nabla\phi$ for some scalar function ϕ defined on D.

Eksempel 12.4.13 Vis at vektorfeltet

s. 641

$$\mathbf{F}(x, y, z) = zy\cos(xy)\,\mathbf{i} + zx\cos(xy)\,\mathbf{j} + \sin(xy)\,\mathbf{k}$$

LHL

er konservativt i $\mathbb{R}^3$, og finn en potensialfunksjon $f(x, y, z)$ for $\mathbf{F}$.

Maxwell's current density equation

In physics, it is shown that if I is the current crossing any surface S bounded by the closed curve C, then

$$\oint_C \mathbf{H} \cdot d\vec{r} = I \quad \text{and} \quad \iint_S \mathbf{J} \cdot \hat{n}\, dS = I$$

where $\mathbf{H}$ is the magnetic intensity, and $\mathbf{J}$ is the electric current density. Use this information to derive Maxwell's current density equation $\operatorname{curl} \mathbf{H} = \mathbf{J}$.