Øving 2 - Fourierrekker I

Jon Vegard Venås

1 First note that $f(x)$ is an even function, which implies that $b_n = 0$ for all n in (3). The coefficient a_0 in (1) is given by

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x = \frac{1}{\pi}\int_{0}^{\pi} f(x)\,\mathrm{d}x = \frac{1}{\pi}\int_{0}^{\pi} \sin(x)\,\mathrm{d}x = \frac{1}{\pi}\left[-\cos x\right]_0^{\pi} = \frac{2}{\pi}.$$

Finally, using the trigonometric identity

$$\sin[(n+1)x] - \sin[(n-1)x] = 2\sin x\cos nx$$

we get

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,\mathrm{d}x = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx\,\mathrm{d}x = \frac{2}{\pi}\int_{0}^{\pi} \sin x\cos nx\,\mathrm{d}x\\
&= \frac{1}{\pi}\int_{0}^{\pi} \sin[(n+1)x] - \sin[(n-1)x]\,\mathrm{d}x\\
&= \frac{1}{\pi}\left[-\frac{\cos[(n+1)x]}{n+1} + \frac{\cos[(n-1)x]}{n-1}\right]_0^{\pi}\\
&= \frac{1}{\pi}\left(-\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right) = -\frac{2}{\pi}\frac{(-1)^n+1}{n^2-1}\\
&= \begin{cases} -\frac{4}{\pi}\frac{1}{n^2-1} & n \text{ even}\\ 0 & n \text{ odd.}\end{cases}
\end{aligned}$$

where we have used $\cos n\pi = (-1)^n$. The Fourier series of $f(x)$ is thus given by

$$f(x) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}\cos 2nx.$$

By evaluating the Fourier series above wisely at $x = 0$ we get

$$f(0) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}.$$

Since we also have $f(0) = 0$, we find

$$\sum_{n=1}^{\infty}\frac{1}{4n^2-1} = \frac{1}{2}$$

2 First note that the function has period 2π. The Fourier coefficients a_n and b_n of a function with period 2π is given by

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x \tag{1}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,\mathrm{d}x \tag{2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,\mathrm{d}x. \tag{3}$$

Using the orthogonality property of trigonometric system, some work can be spared by extracting these coefficients directly from the function:

$$a_0 = 5$$
$$a_n = \begin{cases} -4 & n = 2 \\ 5 & n = 8 \\ 0 & \text{otherwise} \end{cases}$$
$$b_n = \begin{cases} -2 & n = 5 \\ 0 & \text{otherwise.} \end{cases}$$

3 This function is neither odd nor even, and so all coefficients in (1), (2) and (3). We first compute a_0

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x = \frac{1}{2\pi}\int_0^{\pi} x(\pi - x)\,\mathrm{d}x = \frac{1}{2\pi}\left[\frac{\pi}{2}x^2 - \frac{1}{3}x^3\right]_0^{\pi} = \frac{\pi^2}{12}.$$

The following integrals is needed to compute a_n and b_n

$$\int_0^{\pi} x\cos nx\,\mathrm{d}x = \underbrace{\left[\frac{1}{n}x\sin nx\right]_0^{\pi}}_{=0} - \frac{1}{n}\int_0^{\pi}\sin nx\,\mathrm{d}x = \left[\frac{1}{n^2}\cos nx\right]_0^{\pi} = \frac{(-1)^n - 1}{n^2}$$

$$\int_0^{\pi} x\sin nx\,\mathrm{d}x = \left[-\frac{1}{n}x\cos nx\right]_0^{\pi} + \frac{1}{n}\underbrace{\int_0^{\pi}\cos nx\,\mathrm{d}x}_{=0} = -\frac{\pi}{n}(-1)^n$$

$$\int_0^{\pi} x^2\cos nx\,\mathrm{d}x = \underbrace{\left[\frac{1}{n}x^2\sin nx\right]_0^{\pi}}_{=0} - \frac{2}{n}\underbrace{\int_0^{\pi} x\sin nx\,\mathrm{d}x}_{=-\frac{\pi}{n}(-1)^n} = \frac{2\pi}{n^2}(-1)^n$$

$$\int_0^{\pi} x^2\sin nx\,\mathrm{d}x = \left[-\frac{1}{n}x^2\cos nx\right]_0^{\pi} + \frac{2}{n}\underbrace{\int_0^{\pi} x\cos nx\,\mathrm{d}x}_{=\frac{(-1)^n - 1}{n^2}}$$
$$= -\frac{\pi^2}{n}(-1)^n + \frac{2}{n^3}\left[(-1)^n - 1\right].$$

We now have

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,\mathrm{d}x = \frac{1}{\pi}\int_0^{\pi} x(\pi - x)\cos nx\,\mathrm{d}x$$
$$= \int_0^{\pi} x\cos nx\,\mathrm{d}x - \frac{1}{\pi}\int_0^{\pi} x^2\cos nx\,\mathrm{d}x = \frac{(-1)^n - 1}{n^2} - \frac{2}{n^2}(-1)^n$$
$$= \begin{cases} -\frac{2}{n^2} & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,\mathrm{d}x = \frac{1}{\pi}\int_0^{\pi} x(\pi - x)\sin nx\,\mathrm{d}x$$
$$= \int_0^{\pi} x\sin nx\,\mathrm{d}x - \frac{1}{\pi}\int_0^{\pi} x^2\sin nx\,\mathrm{d}x$$
$$= -\frac{\pi}{n}(-1)^n - \frac{1}{\pi}\left[-\frac{\pi^2}{n}(-1)^n + \frac{2}{n^3}\left[(-1)^n - 1\right]\right] = \begin{cases} \frac{4}{n^3\pi} & n \text{ odd} \\ 0 & n \text{ even.} \end{cases}$$

The Fourier series of $f(x)$ is thus

$$f(x) = \frac{\pi^2}{12} - \sum_{n=1}^{\infty} \frac{1}{2n^2} \cos 2nx + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^3} \sin[(2n-1)x]$$

By evaluating the Fourier series above wisely at $x = 0$ we get

$$f(0) = \frac{\pi^2}{12} - \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

Since we also have $f(0) = 0$, we find

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$