TMA4106 Matematikk 2 for MTELSYS

Øving 3 - Fourierrekker II

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1 As f is an even function, $b_n = 0$ for all n. With $L = \frac{1}{2}$ we find

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,\mathrm{d}x = 2\int_0^{\frac{1}{2}} \cos \pi x\,\mathrm{d}x = \left[\frac{1}{\pi}\sin \pi x\right]_0^{\frac{1}{2}} = \frac{2}{\pi}$$

and, using the following formula

$$\cos \pi x \cos 2n\pi x = \frac{1}{2}\left\{\cos[\pi(1+2n)x] + \cos[\pi(1-2n)x]\right\},$$

we have

$$\begin{aligned}
a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,\mathrm{d}x = 4\int_0^{\frac{1}{2}} \cos \pi x \cos 2n\pi x\,\mathrm{d}x \\
&= 2\int_0^{\frac{1}{2}} \cos[\pi(1+2n)x] + \cos[\pi(1-2n)x]\,\mathrm{d}x \\
&= \frac{2}{\pi}\left[\frac{\sin[\pi(1+2n)x]}{1+2n} + \frac{\sin[\pi(1-2n)x]}{1-2n}\right]_0^{\frac{1}{2}} \\
&= \frac{2\cos \pi n}{\pi}\left(\frac{1}{1+2n} + \frac{1}{1-2n}\right) = \frac{4}{\pi}\frac{(-1)^n}{1-4n^2}.
\end{aligned}$$

Thus, the Fourier series of $f(x)$ is

$$f(x) = \frac{2}{\pi} + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{1-4n^2}\cos 2n\pi x.$$

2 The odd periodic extensions of the function $f(x)$ is simply $\sin x$ since the sin-function is odd. The even periodic extensions of the function $f(x)$ is given by $|\sin x|$. A sketch of these functions can be found in Figure 1.

As the odd periodic extension of $f(x)$ is $\sin x$, the Fourier sine series of f will be the Fourier series of $\sin x$. Since $\sin x$ is written on the Fourier series form

$$\sin x = \sum_{n=1}^{\infty} b_n \sin nx = 1\cdot\sin x + 0\cdot\sin 2x + 0\cdot\sin 3x + \ldots$$

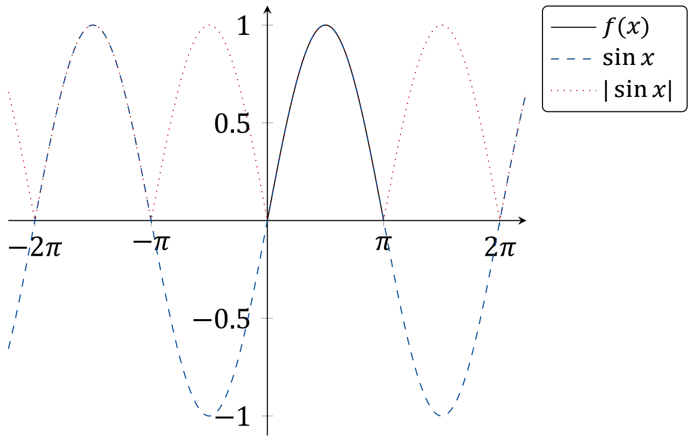
it is also the Fourier series representation of the odd periodic extension of f.

Using the following formula [?, (11) p. A52]

$$\sin x \cos nx = \frac{1}{2}\left\{\sin[(1+n)x] + \sin[(1-n)x]\right\},$$

the coefficients of the Fourier cosine series of f is given by [?, (6*) p. 486]

$$a_0 = \frac{1}{L}\int_0^{L} f(x)\,\mathrm{d}x = \frac{1}{\pi}\int_0^{\pi}\sin x\,\mathrm{d}x = \frac{1}{\pi}\left[-\cos x\right]_0^{\pi} = \frac{2}{\pi}$$



Figur 1: Sketch of the odd ($\sin x$) and even ($|\sin x|$) periodic extensions of $f(x)$.

and for $n > 1$

$$\begin{aligned} a_n &= \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,\mathrm{d}x = \frac{2}{\pi}\int_0^\pi \sin x\cos nx\,\mathrm{d}x \\ &= \frac{1}{\pi}\int_0^\pi \sin[(1+n)x] + \sin[(1-n)x]\,\mathrm{d}x \\ &= \frac{1}{\pi}\left[-\frac{\cos[(1+n)x]}{1+n} - \frac{\cos[(1-n)x]}{1-n}\right]_0^\pi \\ &= \frac{(-1)^n+1}{\pi}\left(\frac{1}{1+n}+\frac{1}{1-n}\right) = \begin{cases} \frac{4}{\pi}\frac{1}{1-n^2} & n \text{ even} \\ 0 & n \text{ odd} \end{cases}. \end{aligned}$$

The special case $n = 1$ must be considered separately

$$a_1 = \frac{1}{\pi}\int_0^\pi \sin x\cos x\,\mathrm{d}x = \frac{1}{2\pi}\int_0^\pi \sin 2x\,\mathrm{d}x = \frac{1}{2\pi}\left[-\frac{1}{2}\cos 2x\right]_0^\pi = 0$$

Thus, the Fourier cosine series of f is given by [?, (5*) p. 486]

$$f(x) = \frac{2}{\pi} + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{1-4n^2}\cos 2nx.$$

3 As f is an even function, $b_n = 0$ for all n. With $L = \pi$ we find

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,\mathrm{d}x = \frac{1}{\pi}\int_0^\pi x^2\,\mathrm{d}x = \frac{1}{\pi}\left[\frac{1}{3}x^3\right]_0^\pi = \frac{\pi^2}{3}$$

and

$$\begin{aligned} a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,\mathrm{d}x = \frac{2}{\pi}\int_0^\pi x^2\cos nx\,\mathrm{d}x \\ &= \frac{2}{\pi}\left(\left[\frac{1}{n}x^2\sin nx\right]_0^\pi - \frac{2}{n}\int_0^\pi x\sin nx\,\mathrm{d}x\right) \\ &= -\frac{4}{n\pi}\left(\left[-\frac{1}{n}x\cos nx\right]_0^\pi + \frac{1}{n}\int_0^\pi \cos nx\,\mathrm{d}x\right) = \frac{4(-1)^n}{n^2}. \end{aligned}$$

Recall Parseval's identity

$$2a_0^2 + \sum_{n=1}^{\infty} \left(a_n^2 + b_n^2\right) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)^2 \,\mathrm{d}x.$$

Thus,

$$\frac{2\pi^4}{9} + \sum_{n=1}^{\infty} \frac{16}{n^4} = \frac{1}{\pi} \int_{-\pi}^{\pi} x^4 \,\mathrm{d}x = \frac{2}{\pi} \int_{0}^{\pi} x^4 \,\mathrm{d}x = \frac{2}{\pi} \left[\frac{1}{5} x^5\right]_0^{\pi} = \frac{2\pi^4}{5},$$

which implies the result

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$