Testeksamen 5

Oppgave 1 En pendel som svinger fritt opphengt i en stang i taket, modelleres av differensiallikningssystemet

$$\dot{x} = y$$
$$\dot{y} = -\sin x$$

Skriv et pythonscript som løser dette numerisk med Heuns metode. Velg steglengde, tidsintervall og startverdier selv.

Heuns metode: Har $\begin{cases} \dot{x} = f(x,y) \\ \dot{y} = g(x,y) \end{cases}$, startverdier x_0 og y_0, steglengde h

$$\Rightarrow \begin{cases} x_{i+1} = x_i + \frac{h}{2}\left(f(x_i,y_i) + f(x^*_{i+1}, y^*_{i+1})\right) \\ y_{i+1} = y_i + \frac{h}{2}\left(g(x_i,y_i) + g(x^*_{i+1}, y^*_{i+1})\right) \end{cases}$$

hvor
$$x^*_{i+1} = x_i + h \cdot f(x_i, y_i)$$
$$y^*_{i+1} = y_i + h \cdot g(x_i, y_i)$$

$$\begin{aligned} \dot{x} &= y = f(x,y) \\ \dot{y} &= -\sin(x) = g(x,y) \end{aligned} \quad \Rightarrow \quad \begin{aligned} x^*_{i+1} &= x_i + h \cdot y_i \\ y^*_{i+1} &= y_i - h \cdot \sin(x) \end{aligned} \qquad \begin{aligned} x_{i+1} &= x_i + \frac{h}{2}(y_i + y^*_{i+1}) \\ y_{i+1} &= y_i - \frac{h}{2}(\sin(x_i) + \sin(x^*_{i+1})) \end{aligned}$$

Oppgave 2

a) Vis at rekken

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n+\sqrt{n}}$$

er betinget konvergerent.

Betinget konvergent: $\sum_{n=0}^{\infty} a_n$ konvergerer, men $\sum_{n=0}^{\infty} |a_n|$ divergerer

Alternerende rekke testen: 1) $|a_{n+1}| \leq |a_n| \quad \forall n$ ✓
2) $\lim_{n\to\infty} a_n = 0$ ✓ $\Rightarrow$ konvergerer!

$$\sum_{n=1}^{\infty} \left| (-1)^{n+1} \frac{1}{n+\sqrt{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}} \geq \sum_{n=1}^{\infty} \frac{1}{2n} = \frac{1}{2} \underbrace{\sum_{n=1}^{\infty} \frac{1}{n}}_{\text{harmoniske rekken}} = \infty$$

$$\frac{1}{n+\sqrt{n}} \geq \frac{1}{n+n}$$

Sammenligningstesten: • Hvis $\sum a_n = \infty$ og $\sum a_n \leq \sum b_n \Rightarrow \sum b_n = \infty$

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$\Rightarrow$ divergerer $\Rightarrow$ betinget konvergens

b) Finn summen til rekken

$$\sum_{n=1}^{\infty} \frac{1}{n3^n}.$$

$$-\ln(1-x) = \int \frac{1}{1-x}\,dx = \int \sum_{n=0}^{\infty} x^n\,dx = \sum_{n=0}^{\infty} \int x^n\,dx = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = \sum_{n=1}^{\infty} \frac{x^n}{n}$$

$$x = \frac{1}{3} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n3^n} = \sum_{n=1}^{\infty} \frac{(1/3)^n}{n} = -\ln\left(1-\frac{1}{3}\right) = -\ln\left(\frac{2}{3}\right)$$

$$= \ln(1) - \ln\left(\frac{2}{3}\right) = \ln\left(\frac{1}{\frac{2}{3}}\right) = \ln\left(\frac{3}{2}\right)$$

Oppgave 3 Finn grenseverdien

$$\lim_{x\to 0} \frac{e^{x^3}-1}{x-\sin x}.$$

Taylorrekker: $e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!} = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots$

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

$$\frac{e^{x^3}-1}{x-\sin(x)} = \frac{\cancel{1} + x^3 + \frac{(x^3)^2}{2!} + \frac{(x^3)^3}{3!} + \dots - \cancel{1}}{\cancel{x} - (\cancel{x} - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots)} = \frac{x^3 + \frac{x^6}{2} + \frac{x^9}{6} + \dots}{\frac{x^3}{3!} - \frac{x^5}{5!} + \dots} \cdot \frac{\frac{1}{x^3}}{\frac{1}{x^3}}$$

$$= \frac{1 + \frac{x^3}{2} + \frac{x^6}{6} + \dots}{\frac{1}{6} - \frac{x^2}{5!} + \dots} \xrightarrow[x\to 0]{} \frac{1}{\frac{1}{6}} = \underline{\underline{6}}$$

Oppgave 4 Se på følgende vektorer i $\mathbb{R}^3$:

$$\mathbf{v}_1 = \begin{pmatrix} 3 \\ -3 \\ -6 \end{pmatrix} \quad \mathbf{v}_2 = \begin{pmatrix} -2 \\ 2 \\ 4 \end{pmatrix} \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ -1 \\ 8 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 4 \\ -9 \\ 3 \end{pmatrix}$$

Er vektorene $\mathbf{v}_1$, $\mathbf{v}_2$ og $\mathbf{v}_3$ lineært uavhengige? Er $\mathbf{b}$ en lineærkombinasjon av $\mathbf{v}_1$, $\mathbf{v}_2$ og $\mathbf{v}_3$?

Lineær uavhengighet: $x_1 \cdot \vec{v_1} + x_2 \cdot \vec{v_2} + x_3 \cdot \vec{v_3} = 0 \iff x_1 = x_2 = x_3 = 0$

$\vec{b}$ lineærkombinasjon: $x_1 \cdot \vec{v_1} + x_2 \cdot \vec{v_2} + x_3 \cdot \vec{v_3} = \vec{b}$ for noen x_1, x_2, x_3

$$x_1 \cdot \begin{bmatrix} 3 \\ -3 \\ -6 \end{bmatrix} + x_2 \cdot \begin{bmatrix} -2 \\ 2 \\ 4 \end{bmatrix} + x_3 \cdot \begin{bmatrix} 1 \\ -1 \\ 8 \end{bmatrix} = \begin{bmatrix} 3 & -2 & 1 \\ -3 & 2 & -1 \\ -6 & 4 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 3 & -2 & 1 & 4 \\ -3 & 2 & -1 & -9 \\ -6 & 4 & 8 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 3 & -2 & 1 & 4 \\ 0 & 0 & 0 & -5 \\ 0 & 0 & 10 & 11 \end{array}\right]$$

$\Rightarrow 0 = -5$ $\ddot{\frown}$ $\Rightarrow \vec{b}$ <u>ikke</u> linearkombinasjon

$\rightarrow 0 = 0 \Rightarrow$ vi får minst én fri variabel

$\Rightarrow$ lineært avhengige

Oppgave 5 Uttrykk

$$\int_0^1 \frac{e^{-x^2} - 1}{x^2}\, dx$$

som en taylorrekke om 0. Hvor mange ledd må du ta med for å approksimere integralet med en feil på under 10^{-2}?

$$e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!} \Rightarrow e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$$

$$\Rightarrow e^{-x^2} - 1 = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!} - 1 = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{n!}$$

$$\Rightarrow \frac{e^{-x^2} - 1}{x^2} = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-2}}{n!} = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2(n-1)}}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{2n}}{(n+1)!}$$

$$\Rightarrow \int_0^1 \frac{e^{-x^2} - 1}{x^2}\, dx \quad \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{2n}}{(n+1)!}\, dx = \sum_{n=0}^{\infty} \int_0^1 \frac{(-1)^{n+1} x^{2n}}{(n+1)!}\, dx$$

$$= \sum_{n=0}^{\infty} \left[\frac{(-1)^{n+1} x^{2n+1}}{(2n+1)(n+1)!} \right]_0^1 = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)(n+1)!}$$

Merk at
$$\left| \int_0^1 \frac{e^{-x^2} - 1}{x^2}\, dx - \sum_{k=0}^{n-1} \frac{(-1)^{k+1}}{(2k+1)(k+1)!} \right| = \left| \sum_{k=n}^{\infty} \frac{(-1)^{k+1}}{(2k+1)(k+1)!} \right| \leq \frac{1}{(2n+1)(n+1)!}$$

$\Rightarrow$ har en feil på $\leq 10^{-2} = \frac{1}{100}$ når $\frac{1}{(2n+1)(n+1)!} \leq 10^{-2}$

$$n = 0 \Rightarrow \frac{1}{(2 \cdot 0 + 1)(0+1)!} = 1 > 10^{-2}$$

$$n = 1 \Rightarrow \frac{1}{(2 \cdot 1 + 1)(1+1)!} = \frac{1}{6} > 10^{-2}$$

$$n = 2 \Rightarrow \frac{1}{(2 \cdot 2 + 1)(2+1)!} = \frac{1}{30} > \frac{1}{100}$$

$$n = 3 \Rightarrow \frac{1}{(2 \cdot 3 + 1)(3+1)!} = \frac{1}{168} < \frac{1}{100}$$

$\Rightarrow$ Ta med leddene for $n = 0, 1, 2,$ og 3 for å få en feil på $< 10^{-2}$

Oppgave 6 La A være følgende matrise:

$$A = \begin{bmatrix} 9 & -3 \\ -3 & 2 \end{bmatrix}$$

Finn alle 2×2-matriser X som er løsninger av likningen $AX = XA$.

$$X = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$$

$$AX = \begin{bmatrix} 9 & -3 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} = \begin{bmatrix} 9x_1 - 3x_3 & 9x_2 - 3x_4 \\ -3x_1 + 2x_3 & -3x_2 + 2x_4 \end{bmatrix}$$

$$\shortparallel$$

$$XA = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}\begin{bmatrix} 9 & -3 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 9x_1 - 3x_2 & -3x_1 + 2x_2 \\ 9x_3 - 3x_4 & -3x_3 + 2x_4 \end{bmatrix}$$

$$\begin{cases} 9x_1 - 3x_3 = 9x_1 - 3x_2 \\ 9x_2 - 3x_4 = -3x_1 + 2x_2 \\ -3x_1 + 2x_3 = 9x_3 - 3x_4 \\ -3x_2 + 2x_4 = -3x_3 + 2x_4 \end{cases} \Rightarrow \begin{cases} 3x_2 - 3x_3 = 0 \\ 3x_1 + 7x_2 - 3x_4 = 0 \\ -3x_1 - 7x_3 + 3x_4 = 0 \\ -3x_2 + 3x_3 = 0 \end{cases}$$

$$\Rightarrow \left[\begin{array}{cccc|c} 0 & 3 & -3 & 0 & 0 \\ 3 & 7 & 0 & -3 & 0 \\ -3 & 0 & -7 & 3 & 0 \\ 0 & -3 & 3 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 0 & 3 & -3 & 0 & 0 \\ 3 & 7 & 0 & -3 & 0 \\ 0 & 7 & -7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 0 & 1 & -1 & 0 & 0 \\ 1 & 7/3 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \begin{cases} x_2 - x_3 = 0 \\ x_1 + \frac{7}{3}x_2 - x_4 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = -\frac{7}{3}x_2 + x_4 \\ x_2 = x_3 \end{cases}$$

La $x_2 = s$ og $x_4 = t$ være frie variabler

$$\Rightarrow x_1 = -\frac{7}{3}s + t, \quad x_2 = s, \quad x_3 = s, \quad x_4 = t$$

$$\Rightarrow X = \begin{bmatrix} -\frac{7}{3}s + t & s \\ s & t \end{bmatrix} = \begin{bmatrix} -\frac{7}{3} & 1 \\ 1 & 0 \end{bmatrix}s + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}t$$

Oppgave 7 Finn buelengden til grafen til $f : [1, 2] \to \mathbb{R}$ gitt ved

$$f(x) = \frac{x^3}{3} + \frac{1}{4x}$$

Buelengde : $\int_a^b \sqrt{1 + (f'(x))^2}\, dx$

$$f'(x) = x^2 - \frac{1}{4x^2}$$

$$\Rightarrow \ell = \int_1^2 \sqrt{1 + \left(x^2 - \frac{1}{4x^2}\right)^2}\, dx = \int_1^2 \sqrt{1 + \left(\frac{4x^4-1}{4x^2}\right)^2}\, dx$$

$$= \int_1^2 \sqrt{\frac{16x^4 + (4x^4-1)^2}{16x^4}}\, dx = \int_1^2 \frac{\sqrt{16x^8 + 8x^4 + 1}}{4x^2}\, dx = \int_1^2 \frac{\sqrt{(4x^4+1)^2}}{4x^2}\, dx$$

$$= \int_1^2 \frac{4x^4+1}{4x^2}\, dx = \int_1^2 x^2 + \frac{1}{4x^2}\, dx = \left[\frac{x^3}{3} - \frac{1}{4x}\right]_1^2 = \frac{8}{3} - \frac{1}{8} - \frac{1}{3} + \frac{1}{4} = \underline{\underline{\frac{59}{24}}}$$

Oppgave 8 Avgjør om integralet

$$\int_0^3 \frac{dx}{(x-2)^{1/3}}$$

konvergerer.

$$= \lim_{b \to 2-} \int_0^b \frac{dx}{(x-2)^{1/3}} + \lim_{a \to 2+} \int_a^3 \frac{dx}{(x-2)^{1/3}}$$

$$= \lim_{b \to 2-} \left[\frac{3}{2}(x-2)^{2/3}\right]_0^b + \lim_{a \to 2+} \left[\frac{3}{2}(x-2)^{2/3}\right]_a^3$$

$$= \lim_{b \to 2-} \frac{3(b-2)^{2/3}}{2} - \frac{3(-2)^{2/3}}{2} + \lim_{a \to 2+} \frac{3}{2} - \frac{3(a-2)^{2/3}}{2}$$

$$= 0 - \frac{3 \cdot 2^{2/3}}{2} + \frac{3}{2} - 0 = \frac{3}{2}\left(1 - 2^{2/3}\right)$$

$$(-2)^{2/3} = (-1 \cdot 2)^{2/3}$$
$$= (-1)^{2/3} \cdot 2^{2/3}$$
$$= \left((-1)^2\right)^{1/3} \cdot 2^{2/3}$$
$$= 1 \cdot 2^{2/3}$$

Oppgave 9 Gjør rede for at $f : [-1, 1] \to \mathbb{R}$ gitt ved

$$f(x) = \ln(\arcsin(x) + 2)$$

har en invers funksjon, og finn denne.

$$f'(x) = \frac{1}{\arcsin(x)+2} \cdot \frac{1}{\sqrt{1-x^2}} > 0 \quad \text{for } x \in (-1, 1)$$

$\Rightarrow f$ monotont stigende

$\Rightarrow f$ injektiv

$\Rightarrow f^{-1}$ finnes

$$f(y) = \ln(\arcsin(y) + 2) = x$$

$$\Rightarrow \arcsin(y) + 2 = e^x$$

$\Rightarrow \arcsin(y) = e^x - 2$

$\Rightarrow y = \sin(e^x - 2)$

$\Rightarrow f^{-1}(x) = \sin(e^x - 2)$