Komplekse tall

$\sqrt{-1} =: i$

1 Finn alle løsninger av likningen $x^2 + x + 1 = 0$.

$$x = \frac{-1 \pm \sqrt{1^2 - 4\cdot 1 \cdot 1}}{2\cdot 1} = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm \sqrt{3}\sqrt{-1}}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$$

2 Faktoriser polynomet $x^2 + 2x + 2$.

$$x = \frac{-2 \pm \sqrt{2^2 - 4\cdot 1 \cdot 2}}{2\cdot 1} = \frac{-2 \pm \sqrt{-4}}{2} = \frac{-2 \pm 2i}{2} = -1 \pm i$$

$$x^2 + 2x + 2 = (x - (-1+i))(x - (-1-i))$$

Kartesisk form: $z = a + bi \qquad a, b \in \mathbb{R}$

Realdelen: $\mathrm{Re}(z) = a$ Imaginærdelen: $\mathrm{Im}(z) = b$

$$z = \mathrm{Re}(z) + \mathrm{Im}(z)i$$

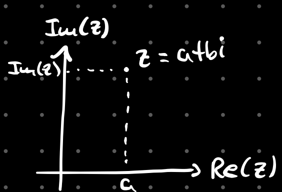
Mengden av komplekse tall: $\mathbb{C}$ $\qquad z \in \mathbb{C}$

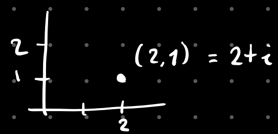
Merk: $\mathbb{R} \subseteq \mathbb{C}$ (for z sånn at $b = 0$)

Komplekse planet

$\mathbb{C} \longleftrightarrow \mathbb{R}^2$

$a + bi \longleftrightarrow (a, b)$





Regneregler for kartesisk form

La $z = a + bi$ og $w = c + di$ være komplekse tall. Vi har

$$z + w = a + c + (b + d)i$$
$$z - w = a - c + (b - d)i$$
$$z \cdot w = ac - bd + (bc + ad)i = (a+bi)(c+di)$$
$$\frac{z}{w} = \frac{ac + bd + (bc - ad)i}{c^2 + d^2} = \frac{z\overline{w}}{w\overline{w}}$$

$(a,b) + (c,d) = (a+c, b+d)$

Konjugert : $z = a+bi \Rightarrow \bar{z} = a-bi$

$z\bar{z} = (a+bi)(a-bi) = a^2+b^2 \in \mathbb{R}$

$\boxed{3}$ La $z = 2+3i$ og $w = 4+5i$. Regn ut $z+w$, $z-w$, $z \cdot w$ og z/w.

$z+w = (2+3i)+(4+5i) = (2+4)+(3+5)i = 6+8i$

$z-w = (2+3i)-(4+5i) = (2-4)+(3-5)i = -2-2i$

$zw = (2+3i)(4+5i) = 8+10i+12i-15 = -7+22i$

$$\frac{z}{w} = \frac{z\bar{w}}{w\bar{w}} = \frac{(2+3i)(4-5i)}{(4+5i)(4-5i)} = \frac{8-10i+12i+15}{4^2+5^2} = \frac{23+2i}{16+25} = \frac{23+2i}{41} = \frac{23}{41}+\frac{2}{41}i$$

$\boxed{4}$ La $z = a+bi$ og $\underset{\substack{\parallel \\ c+di}}{w}$ være komplekse tall. Vis at:

$$\overline{z+w} = \bar{z}+\bar{w} \qquad \overline{z-w} = \bar{z}-\bar{w}$$
$$\overline{z\cdot w} = \bar{z}\cdot\bar{w} \qquad \overline{z/w} = \bar{z}/\bar{w}$$
$$z+\bar{z} = 2a \qquad z-\bar{z} = 2bi$$

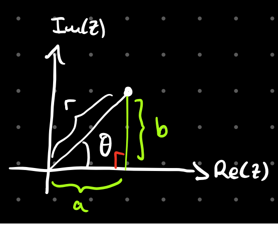
$\overline{z+w} = \overline{(a+bi)+(c+di)} = \overline{(a+c)+(b+d)i} = (a+c)-(b+d)i$

$\overline{z-w} = \bar{z}-\bar{w}$, $\overline{zw} = \bar{z}\cdot\bar{w}$, $\overline{\left(\frac{z}{w}\right)} = \frac{\bar{z}}{\bar{w}}$ lignende

$z+\bar{z} = (a+bi)+(a-bi) = 2a = 2\,\mathrm{Re}(z)$

$z-\bar{z} = 2bi = 2\,\mathrm{Im}(z)$

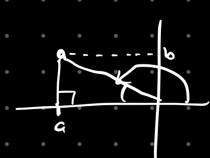
Polarkoordinater: $(z \neq 0)$



$\boxed{5}$ Gjør rede for at

$$a = \mathrm{Re}(z) = r\cos\theta \qquad b = \mathrm{Im}(z) = r\sin\theta \qquad r = \sqrt{a^2+b^2}$$

$$\theta = \begin{cases} \arctan\left(\frac{b}{a}\right) & a>0 \\ \arctan\left(\frac{b}{a}\right)+\pi & a<0 \\ \pi/2 & a=0,\ b>0 \\ 3\pi/2 & a=0,\ b<0 \end{cases}$$



$\tan\left(\frac{b}{a}\right)$

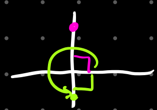
$$\cos(\theta) = \frac{a}{r} \Rightarrow a = r\cos\theta$$

$$\sin(\theta) = \frac{b}{r} \Rightarrow b = r\sin\theta$$

$$a^2 + b^2 = r^2 \Rightarrow r = \sqrt{a^2+b^2}$$

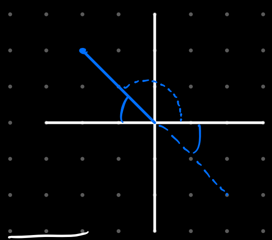
$$\tan(\theta) = \frac{b}{a} \Rightarrow \theta = \arctan\left(\frac{b}{a}\right) \quad \text{for } a>0$$

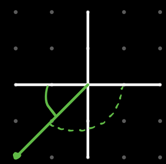
Når $a<0 \Rightarrow \theta = \arctan\left(\frac{b}{a}\right) + \pi$ [slå opp!]



$a<0, b>0 \Rightarrow \theta = \frac{\pi}{2}$

$a<0, b<0 \Rightarrow \theta = \frac{3\pi}{2} = -\frac{\pi}{2}$





$$\theta = \begin{cases} \arctan\left(\frac{b}{a}\right) & a>0 \\ \arctan\left(\frac{b}{a}\right)+\pi & a<0,\ b\geq 0 \\ \arctan\left(\frac{b}{a}\right)-\pi & a<0,\ b<0 \\ \frac{\pi}{2} & a=0,\ b>0 \\ \frac{3\pi}{2} = -\frac{\pi}{2} & a=0,\ b<0 \end{cases}$$

$\Rightarrow$ polarform : $z = re^{i\theta}$ $\quad r \geq 0, \quad 0 \leq \theta < 2\pi$

Modulus / absoluttverdi : $z = a+bi$ $\quad |z| = \sqrt{a^2+b^2} = r = \sqrt{z\bar{z}}$ $\quad (z\bar{z} = r^2)$

Argumentet : $\arg(z) = \theta$

6 Skriv opp taylorrekkene for e^x, $\cos x$ og $\sin x$. Skriv så opp taylorrekken for e^{ix} ved å substituere ix for x i taylorrekken til e^x, og nistirr på denne en lang stund.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$

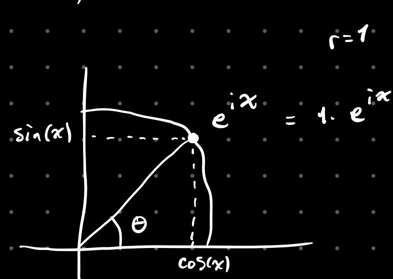
$$e^{ix} = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} \dots$$

$$= 1 + \frac{i^2x^2}{2!} + \frac{i^4x^4}{4!} + \dots + ix + \frac{i^3x^3}{3!} + \frac{i^5x^5}{5!} \dots$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right)$$

$$= \cos(x) + i\sin(x)$$

$\Rightarrow$ $\boxed{\text{Eulers formel : } e^{ix} = \cos(x) + i\sin(x)}$



Alternativt:

- Definer $f(x) = \frac{\cos(x) + i\sin(x)}{e^{ix}} = e^{-ix}(\cos(x) + i\sin(x))$ for $x \in \mathbb{R}$

$$\Rightarrow f'(x) = e^{-ix}(i\cos(x) - \sin(x)) - ie^{-ix}(\cos(x) + i\sin(x))$$

$$= e^{-ix}(i\cos(x) - \sin(x) - i\cos(x) + \sin(x)) = 0$$

$\Rightarrow f(x)$ er konstant

$$f(0) = \frac{\cos(0) + i\cdot\sin(0)}{e^{i\cdot 0}} = \frac{1+0}{1} = 1$$

$\Rightarrow f(x) = 1 \qquad \Rightarrow e^{ix} = \cos(x) + i\sin(x)$

- e^{ix} er kompleks, så $e^{ix} = a+bi = r(\cos\theta + i\sin\theta)$ for noen $r(x)$ og $\theta(x)$

$$\Rightarrow ie^{ix} = \frac{dr}{dx}(\cos\theta + i\sin\theta) + r(-\sin\theta + i\cos\theta)\frac{d\theta}{dx}$$

$$\parallel$$

$$ir(\cos\theta + i\sin\theta) = r(i\cos\theta - \sin\theta)$$

$\Rightarrow \frac{dr}{dx} = 0$ og $\frac{d\theta}{dx} = 1$

$\Rightarrow r(x)$ er konstant og $\theta(x) = x + c$ $\quad c$ konst.

$x = 0 \Rightarrow e^{i0} = 1 \cdot e^{i\cdot 0}$

$r(0) = 1 \qquad \theta(0) = 0 = 0 + c \Rightarrow c = 0$

$\Rightarrow r = 1$ og $\theta = x$

$$\Rightarrow e^{ix} = 1\cdot(\cos\theta + i\sin\theta) = \cos\theta + i\sin\theta$$

$\boxed{7}$ Bruk trigonometriske relasjoner til å vise at $e^{i(x+y)} = e^{ix}e^{iy}$.
(Denne er mest som ekstra sikkerhet for de som ikke liker rekker.)

$$e^{i(x+y)} = \cos(x+y) + i\sin(x+y) = \underline{\cos(x)\cos(y)} - \underline{\sin(x)\sin(y)} + i\,\underline{\sin(x)\cos(y)} + i\,\underline{\cos(x)\sin(y)}$$

$$e^{ix}e^{iy} = (\cos(x) + i\sin(x))(\cos(y) + i\sin(y)) = \underline{\cos(x)\cos(y)} + \underline{i\cos(x)\sin(y)} + \underline{i\sin(x)\cos(y)} - \underline{\sin(x)\sin(y)}$$

Med polarform: $z\cdot w = |z|e^{i\cdot\arg(z)} \cdot |w|e^{i\cdot\arg(w)} = |z|\cdot|w|\,e^{i(\arg(z) + \arg(w))}$

$$\frac{z}{w} = \frac{|z|e^{i\cdot\arg(z)}}{|w|e^{i\cdot\arg(z)}} = \frac{|z|}{|w|}e^{i(\arg(z) - \arg(w))}$$

8 La z og w være følgende komplekse tall:

$$z = \frac{3\pi}{4}i \qquad w = -\frac{3\pi}{4}i$$

Skriv tallene $e^z - e^w$ og e^z/e^w på polar form.

cos er jevn

sin er odde

$$e^z - e^w = e^{\frac{3\pi}{4}i} - e^{-\frac{3\pi}{4}i} = \cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right) - \cos\left(+\frac{3\pi}{4}\right) + i\sin\left(+\frac{3\pi}{4}\right)$$

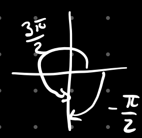
$$= 2\cdot i\cdot\sin\left(\frac{3\pi}{4}\right) = 2\cdot\frac{\sqrt{2}}{2}\cdot i = \sqrt{2}\,i$$

$$a=0,\ b=\sqrt{2} \implies r = \sqrt{0^2+\sqrt{2}^2} = \sqrt{2}$$

$$\theta = \frac{\pi}{2}$$

$$\implies e^z - e^w = \sqrt{2}\,e^{i\frac{\pi}{2}}$$

$$\frac{e^z}{e^w} = \frac{e^{\frac{3\pi}{4}i}}{e^{-\frac{3\pi}{4}i}} = e^{\frac{3\pi}{4}i - \left(-\frac{3\pi}{4}i\right)} = e^{\frac{6\pi}{4}i} = e^{\frac{3\pi}{2}i} = e^{-\frac{\pi}{2}i}$$



9 Vis at dersom $z = re^{i\theta}$ gir Eulers formel $\overline{z} = re^{-i\theta}$.

$$\overline{a+bi} = a-bi$$

$$\overline{z} = \overline{re^{i\theta}} = \overline{r}\,\overline{e^{i\theta}} = r\,\overline{(\cos\theta + i\sin\theta)} = r(\cos\theta - i\sin\theta)$$

$$= r(\cos(-\theta) + i\sin(-\theta)) = re^{-i\theta}$$