Funksjoner: $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$
$\mathbb{R}\times\mathbb{R}$

Fram til nå: $f: \mathbb{R} \to \mathbb{R}$, $f(x) = \dots \leftarrow$ uttrykk med x

Nå: $f: \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = \dots \leftarrow$ uttrykk med x og y

e.g $f(x,y) = x^3 + xy^2 + 1$

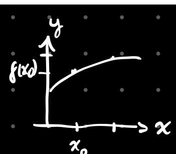
Visualisering

$\boxed{13}$ Skisser nivåkurvene til

$$f(x,y) = x + 2y$$

Akkurat som på kart, kan man bruke avstanden mellom ekvidistanselinjene til å indikere hvor bratt funksjonen stiger.

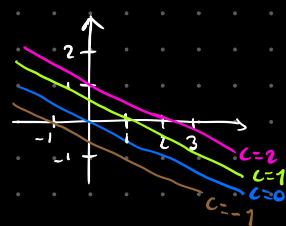
Plotting: $f(x) = y \rightsquigarrow$

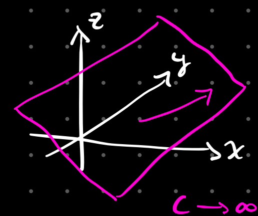


$f(x,y) = z \rightsquigarrow$

Nivåkurver: Se på $f(x,y) = c$ for c konstant

$f(x,y) = x + 2y = c \implies y = \frac{1}{2}(c - x)$





$\boxed{17}$ Skisser nivåkurvene til

$$g(x,y) = (x^2 + 2y)^2$$

Man kan bruke python til å plotte nivåkurver:
https://matplotlib.org/stable/gallery/images_contours_and_fields/contour_demo.html

$\boxed{31}$ Plot nivåkurvene til f og g i python.

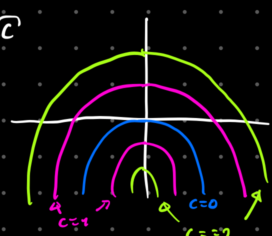
$g(x,y) = (x^2 + 2y)^2 = c \implies x^2 + 2y = \pm\sqrt{c}$

$\implies y = \frac{1}{2}(\pm\sqrt{c} - x^2)$

$c = 0 \implies y = -\frac{x^2}{2}$

$c = 1 \implies y = \frac{1}{2}(1 - x^2)$ og $y = \frac{1}{2}(-1 - x^2)$

~~$c = -1 \implies y =$ kompleks!~~



Derivasjon / partiell derivasjon

$f(x) \rightsquigarrow f'(x) = \frac{df}{dx} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$f(x,y) \rightsquigarrow \frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}$ = "f derivert mhp x"

$\frac{\partial f}{\partial y} = \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h}$ = "f derivert mhp y"

Andre notasjoner: $\partial_x f$, f_x, ...

I praksis:
- $\frac{\partial f}{\partial x}$: lat som y er konstant, deriver mhp x
- $\frac{\partial f}{\partial y}$: lat som x er konstant, deriver mhp y

$f(x,y) = xy \Rightarrow \frac{\partial f}{\partial x} = y, \quad \frac{\partial f}{\partial y} = x$

79 Finn de partiellderiverte til $f(x,y) = x^2 + xy + y^2 + x + y$.

97 Finn de partiellderiverte til

$$f(x,y) = \frac{x}{\sqrt{x^2+y^2}}$$

79 $\frac{\partial f}{\partial x} = 2x + y + 0 + 1 + 0 = 2x + y + 1$

$\frac{\partial f}{\partial y} = 0 + x + 2y + 0 + 1 = x + 2y + 1$

97 $\frac{\partial f}{\partial x} = \frac{1\cdot(x^2+y^2)^{1/2} - x\cdot\frac{1}{2}\cdot(x^2+y^2)^{-\frac{1}{2}}\cdot 2x}{\sqrt{x^2+y^2}^2}$

$(x^2+y^2)^{1/2} = u^{\frac{1}{2}}$
$\Rightarrow (u^{\frac{1}{2}})' = \frac{1}{2}u^{-\frac{1}{2}}\cdot u'$

$= \frac{1}{x^2+y^2}\left(\frac{x^2+y^2}{(x^2+y^2)^{1/2}} - \frac{x^2}{(x^2+y^2)^{1/2}}\right) = \frac{y^2}{(x^2+y^2)(x^2+y^2)^{1/2}} = \frac{y^2}{(x^2+y^2)^{3/2}}$

$\frac{\partial f}{\partial y} = \frac{0\cdot\sqrt{\dots} - x\cdot\frac{1}{2}(x^2+y^2)^{-\frac{1}{2}}\cdot 2y}{x^2+y^2} = \frac{-xy}{(x^2+y^2)(x^2+y^2)^{1/2}} = \frac{-xy}{(x^2+y^2)^{3/2}}$

Det er vanlig å sette opp der partiellderiverte i en vektor, kalt gradientvektoren:

$$\nabla f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right]$$

107 Finn gradientvektoren til $f(x,y) = x^2 + xy + y^2 + x + y$.

113 Finn gradientvektoren til

$$f(x,y) = \frac{1}{\sqrt{x^2+y^2}}$$

149 Finn gradientvektoren til $g(x,y) = \sin x \sin y$.

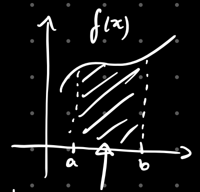
107 $\nabla f = [2x+y+1,\ x+2y+1]$

∇ = "nabla"

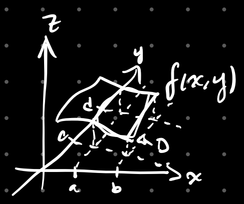
$\boxed{149}$ $\frac{\partial g}{\partial x} = \cos(x)\sin(y)$, $\frac{\partial g}{\partial y} = \sin(x)\cos(y) \Rightarrow \nabla g = [\cos(x)\sin(y),\ \sin(x)\cos(y)]$

<u>Integrasjon</u>

Geometrisk :



$\int_a^b f(x)\,dx$ = Arealet



$\iint_D f(x,y)\,dx\,dy$ = Volumet

↳ D = "grunnflaten"

$\boxed{347}$ Beregn

$$\int_0^2 \left(\int_0^1 (x+2y)\,dx\right) dy,$$

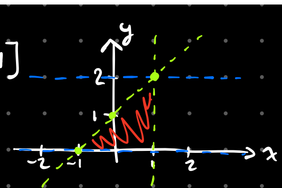
$$\int_0^2\int_0^1 x+2y\,dx\,dy = \int_0^2 \left[\frac{x^2}{2} + 2xy\right]_{x=0}^{x=1} dy = \int_0^2 \frac{1}{2} + 2y\,dy = \left[\frac{y}{2} + y^2\right]_{y=0}^{y=2} = \frac{2}{2} + 2^2 = 1+4 = \underline{\underline{5}}$$

$\boxed{359}$ Skisser integrasjonsområdet til

$$\int_0^2 \int_{y-1}^1 (x+2y)\,dx\,dy,$$

og regn ut det itererte integralet.

$y \in [0,2]$, $x \in [y-1, 1]$



$$\int_0^2\int_{y-1}^1 x+2y\,dx\,dy = \int_0^2 \left[\frac{x^2}{2} + 2xy\right]_{x=y-1}^{x=1} dy = \int_0^2 \frac{1}{2} + 2y - \frac{(y-1)^2}{2} - 2(y-1)y\,dy$$

$$= \int_0^2 \cancel{\frac{1}{2}} + 2y - \frac{y^2}{2} + \cancel{\frac{2y}{2}} - \cancel{\frac{1}{2}} - 2y^2 + 2y\,dy = \int_0^2 5y - \frac{5}{2}y^2\,dy$$

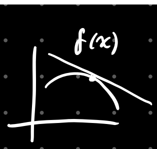
$$= \left[\frac{5}{2}y^2 - \frac{5}{6}y^3\right]_{y=0}^{y=2} = \frac{5}{2}\cdot 4 - \frac{5}{6}\cdot 8 = 10 - \frac{20}{3} = \frac{30}{3} - \frac{20}{3} = \underline{\underline{\frac{10}{3}}}$$

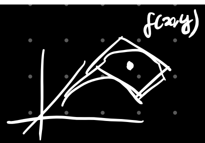
Tangentplanet til f i punktet (x_0, y_0) er gitt ved

$$z = a(x - x_0) + b(y - y_0) + f(x_0, y_0)$$

der

$$a = \frac{\partial f}{\partial x}(x_0, y_0) \quad \text{og} \quad b = \frac{\partial f}{\partial y}(x_0, y_0)$$





$\boxed{167}$ Finn tangentplanet til $f(x,y) = x^2 + xy + y^2 + x + y$ i $(1,2)$.

$$\frac{\partial f}{\partial x} = 2x + y + 1 \,, \qquad \frac{\partial f}{\partial y} = x + 2y + 1$$

$$\frac{\partial f}{\partial x}(1,2) = 2\cdot 1 + 2 + 1 = 5 \,, \quad \frac{\partial f}{\partial y}(1,2) = 1 + 2\cdot 2 + 1 = 6 \,, \quad f(1,2) = 1 + 2 + 4 + 1 + 2 = 10$$

$$\Rightarrow \; z = 5(x-1) + 6(y-2) + 10 = 5x - 5 + 6y - 12 + 10 = 5x + 6y - 7$$