Determinants

Two by two matrices

$$\det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad-bc$$

Three by three matrices

$$\det A = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} =$$

$$= a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1$$

General definition

Suppose that determinants of $(n-1)$ by $(n-1)$ matrices are already defined.
Let A be an n by n matrix, $A=[a_{ij}]$.
The ijth minor of A is the determinant of $(n-1)$ by $(n-1)$ matrix that remains after deleting the ith row and jth column of A. It is denoted by M_{ij}.

$A_{ij} = (-1)^{i+j} M_{ij}$ is called the ijth cofactor of A.

$$\det A = a_{11}A_{11} + a_{12}A_{12} + \ldots + a_{1n}A_{1n}$$

Theorem The determinant of an n by n matrix can be obtained by expansion along any row or column

$\det A = a_{i1}A_{i1} + a_{i2}A_{i2} + \ldots + a_{in}A_{in}$ or

$\det A = a_{1j}A_{1j} + a_{2j}A_{2j} + \ldots + a_{nj}A_{nj}$