Orthogonal Projection and Least Square Solution

Theorem Let V be a subspace of R^n and $V^{\perp}$ be its orthogonal complement. Then for any vector $\boldsymbol{b}$ in R^n there exists a unique decomposition $\boldsymbol{b}=\boldsymbol{p}+\boldsymbol{q}$, where $\boldsymbol{p}$ is a vector in V and $\boldsymbol{q}$ is a vector in $V^{\perp}$.

Definition In the above decomposition $\boldsymbol{p}$ is called the orthogonal projection of $\boldsymbol{b}$ onto V.

Assume that the system $\mathbf{A}\boldsymbol{x}=\boldsymbol{b}$ has no solution. It means that $\boldsymbol{b}$ does not lie in the column space of $\mathbf{A}$. Let $V=\mathrm{Col}(\mathbf{A})$, and let $\boldsymbol{p}$ be the orthogonal projection of b onto V.

The least square solution of $\mathbf{A}\boldsymbol{x}=\boldsymbol{b}$ is the vector $\boldsymbol{x}$ for each $\|\mathbf{A}\boldsymbol{x}-\boldsymbol{b}\|^2$ is the least possible. Then $\mathbf{A}\boldsymbol{x}=\boldsymbol{p}$.

To find the least square solution of the system $\mathbf{A}\boldsymbol{x}=\boldsymbol{b}$, we consider the normal system associated with that one: $\mathbf{A}^T\mathbf{A}\boldsymbol{x}=\mathbf{A}^T\boldsymbol{b}$. Solution of the normal system is the least square solution of the initial system.

Matrix $\mathbf{A}^T\mathbf{A}$ is a square matrix and if $\mathbf{A}$ is an $m\times n$ matrix with $\mathrm{rank}(\mathbf{A})=n$, then $\mathbf{A}^T\mathbf{A}$ is non-singular.