Orthogonal Basis and Gram-Schmidt Algorithm

Definition An orthogonal basis for the space V is a basis $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ for which $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ when $i \neq j$.

If in addition $\mathbf{v}_i \cdot \mathbf{v}_i = 1$ for each $i=1,\ldots,k$, then the basis is called an orthonormal basis.

Suppose that V has an orthonormal basis $\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_k$. Then the projection $\mathbf{p}$ of a vector $\mathbf{b}$ onto V is given by the formula

$$\mathbf{p} = \mathbf{v}_1 \cdot \mathbf{b}\, \mathbf{v}_1 + \mathbf{v}_2 \cdot \mathbf{b}\, \mathbf{v}_2 + \ldots + \mathbf{v}_k \cdot \mathbf{b}\, \mathbf{v}_k$$

The Gram-Schmidt Algorithm

Suppose that $\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_k$ is a basis of a subspace V of R^n, to find an orthogonal basis $\mathbf{u}_1, \mathbf{u}_2, \ldots \mathbf{u}_k$ for V we can use the following algorithm:

$$\mathbf{u}_1 = \mathbf{v}_1,$$

$$\mathbf{u}_2 = \mathbf{v}_2 - \frac{\mathbf{u}_1 \cdot \mathbf{v}_2}{\mathbf{u}_1 \cdot \mathbf{u}_1}\, \mathbf{u}_1,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\mathbf{u}_k = \mathbf{v}_k - \frac{\mathbf{u}_1 \cdot \mathbf{v}_k}{\mathbf{u}_1 \cdot \mathbf{u}_1}\, \mathbf{u}_1 - \frac{\mathbf{u}_2 \cdot \mathbf{v}_k}{\mathbf{u}_2 \cdot \mathbf{u}_2}\, \mathbf{u}_2 - \ldots - \frac{\mathbf{u}_{k-1} \cdot \mathbf{v}_k}{\mathbf{u}_{k-1} \cdot \mathbf{u}_{k-1}}\, \mathbf{u}_{k-1}.$$