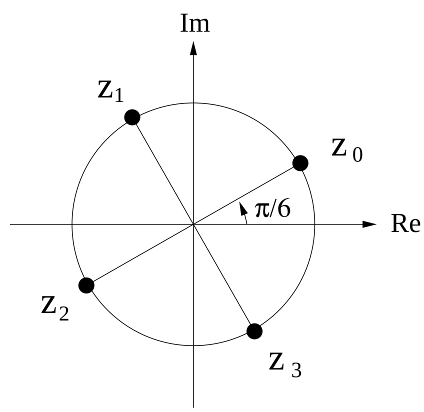
Løsningsforslag
Eksamen SIF5009, desember 2002

Oppgave 1

På polarform har vi $z^4 = 16\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right) = 16e^{\frac{2\pi}{3}i}$,
så $z_k = 2e^{i\left(\frac{2\pi}{3}+2k\pi\right)/4}$ for $k = 0, 1, 2, 3$, dvs.

$$\begin{aligned} z_0 &= 2e^{\frac{\pi}{6}i} = \sqrt{3}+i \\ z_1 &= 2e^{\frac{2\pi}{3}i} = -1+\sqrt{3}i \\ z_2 &= 2e^{\frac{7\pi}{6}i} = -\sqrt{3}-i \\ \underline{\underline{z_3}} &\underline{\underline{= 2e^{\frac{5\pi}{3}i}}} = 1-\sqrt{3}i. \end{aligned}$$



Oppgave 2

a) For $x > 0$ har vi

$$y' + \frac{4}{x}y = 8x^3,$$

og en integrerende faktor er $F(x) = e^{\int \frac{4}{x}dx} = e^{4\ln x} = x^4$. Dette gir

$$(x^4y)' = 8x^7,$$

og da er $x^4y = \int 8x^7dx = x^8 + c$. Initialverdien $y(1) = 2$ gir $c = 1$, så

$$\underline{\underline{y = x^4 + x^{-4},\ x > 0.}}$$

b) Den karakteristiske ligningen $\lambda^2 + 4\lambda + 5 = 0$ har røttene $\lambda = -2 \pm i$, så generell løsning er $y = c_1 e^{-2x} \cos x + c_2 e^{-2x} \sin x$.

Siden $y' = -2c_1 e^{-2x} \cos x - (c_1 + 2c_2)e^{-2x} \sin x + c_2 e^{-2x} \cos x$ gir initialverdiene

$$\begin{aligned} 1 &= y(0) = c_1 \\ 0 &= y'(0) = -2c_1 + c_2, \end{aligned}$$

dvs. $c_1 = 1$ og $c_2 = 2$. Så

$$\underline{\underline{y = e^{-2x} \cos x + 2e^{-2x} \sin x.}}$$

c) Løser først $y'' - 3y' + 2y = 0$. Her gir $\lambda^2 - 3\lambda + 2 = 0$ at $\lambda = 1$ eller 2, så generell løsning er

$$y_h = c_1 e^x + c_2 e^{2x}.$$

Siden 0 ikke er og 1 er løsning av den karakteristiske ligningen, lar vi

$$y_p = A + Bx + Cxe^x.$$

Dette gir

$$\begin{array}{r|lcl} 2 & y & = & A + Bx + Cxe^x \\ -3 & y' & = & B + Cxe^x + Ce^x \\ 1 & y'' & = & \quad\;\; Cxe^x + 2Ce^x \\ \hline & 4x + e^x & = & 2A - 3B + 2Bx - Ce^x, \end{array}$$

og $A = 3$, $B = 2$, $C = -1$, dvs.

$$y_p = 3 + 2x - xe^x.$$

Generell løsning er da $y = y_h + y_p = \underline{\underline{c_1 e^x + c_2 e^{2x} + 3 + 2x - xe^x}}$.

d) Løser først $y'' + 6y' + 9y = 0$. Her gir $\lambda^2 + 6\lambda + 9 = 0$ at $\lambda = -3$ er en dobbelrot, så generell løsning er

$$y_h = c_1 e^{-3x} + c_2 x e^{-3x}.$$

La $y_1 = e^{-3x}$ og $y_2 = xe^{-3x}$, og vi setter

$$y_p = u_1 y_1 + u_2 y_2$$

der u_1 og u_2 oppfyller

$$\begin{aligned} y_1 u_1' + y_2 u_2' &= 0 \\ y_1' u_1' + y_2' u_2' &= \frac{e^{-3x}}{1+x^2}. \end{aligned}$$

Her er

$$W = \begin{vmatrix} e^{-3x} & xe^{-3x} \\ -3e^{-3x} & (1-3x)e^{-3x} \end{vmatrix} = e^{-6x},$$

så Cramers regel gir

$$u_1' = \frac{\begin{vmatrix} 0 & xe^{-3x} \\ \frac{e^{-3x}}{1+x^2} & * \end{vmatrix}}{W} = -\frac{x}{1+x^2} \quad \Rightarrow \quad u_1 = -\frac{1}{2}\ln(1+x^2)$$

og

$$u_2' = \frac{\begin{vmatrix} e^{-3x} & 0 \\ * & \frac{e^{-3x}}{1+x^2} \end{vmatrix}}{W} = \frac{1}{1+x^2} \quad \Rightarrow \quad u_2 = \arctan x.$$

Dermed blir

$$y_p = -\frac{e^{-3x}}{2}\ln(1+x^2) + xe^{-3x}\arctan x.$$

Generell løsning er da

$$y = y_h + y_p = \underline{\underline{c_1 e^{-3x} + c_2 x e^{-3x} - \frac{e^{-3x}}{2}\ln(1+x^2) + xe^{-3x}\arctan x.}}$$

Alternativt: Bruk formel (Kreyszig s. 108)

$$y_p = -y_1 \int \frac{y_2 r}{W} dx + y_2 \int \frac{y_1 r}{W} dx,$$

der y_1, y_2, W er definert ovenfor og $r = e^{-3x}/(1+x^2)$. Det gir

$$\begin{aligned} y_p &= -\, e^{-3x} \int \frac{x}{1+x^2} dx + xe^{-3x} \int \frac{1}{1+x^2} dx \\ &= -\,\frac{e^{-3x}}{2}\ln(1+x^2) + xe^{-3x}\arctan x. \end{aligned}$$

Oppgave 3

a) Gausseliminsajon gir

$$\left[\begin{array}{rrrrr|r} 1 & -2 & 2 & -1 & 2 & 3 \\ 2 & -4 & 1 & 1 & 3 & 1 \\ 1 & -2 & 5 & -4 & 4 & 7 \\ 2 & -4 & 3 & -1 & -1 & 9 \end{array}\right] \sim \left[\begin{array}{rrrrr|r} 1 & -2 & 2 & -1 & 2 & 3 \\ 0 & 0 & -3 & 3 & -1 & -5 \\ 0 & 0 & 3 & -3 & 2 & 4 \\ 0 & 0 & -1 & 1 & -5 & 3 \end{array}\right] \sim$$

$$\left[\begin{array}{rrrrr|r} 1 & -2 & 0 & 1 & -8 & 9 \\ 0 & 0 & 0 & 0 & 14 & -14 \\ 0 & 0 & 0 & 0 & -13 & 13 \\ 0 & 0 & 1 & -1 & 5 & -3 \end{array}\right] \sim \left[\begin{array}{rrrrr|r} 1 & -2 & 0 & 1 & -8 & 9 \\ 0 & 0 & 1 & -1 & 5 & -3 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \sim$$

$$\left[\begin{array}{rrrrr|r} 1 & -2 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} x_1 = 1 + 2s - t \\ x_2 = s \\ x_3 = 2 + t \\ x_4 = t \\ x_5 = -1 \end{array},$$

$$\text{dvs. } \mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \\ -1 \end{bmatrix} + s \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}; \quad s, t \in \mathbb{R}.$$

b) Basis er for

$$\text{Null}(A): \quad \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \quad \text{(f.eks.)}$$

$$\text{Col}(A): \quad \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 5 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \\ -1 \end{bmatrix} \quad \text{(f.eks.)}$$

$$\text{Row}(A): \quad \begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{(f.eks.)}$$

c) Vi bruker Gram-Schmidt på basisen

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Det gir en ortogonal basis $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ der

$$\mathbf{u}_1 = \mathbf{v}_1 = \underline{\underline{\begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}}},$$

$$\mathbf{u}_2 = \mathbf{v}_2 - \left(\frac{\mathbf{v}_2 \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\right)\mathbf{u}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} - \frac{-1}{6}\begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \underline{\underline{\frac{1}{6}\begin{bmatrix} 1 \\ -2 \\ 6 \\ -5 \\ 0 \end{bmatrix}}},$$

$$\mathbf{u}_3 = \mathbf{v}_3 - \left(\frac{\mathbf{v}_3 \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\right)\mathbf{u}_1 - \left(\frac{\mathbf{v}_3 \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2}\right)\mathbf{u}_2 = \mathbf{v}_3 = \underline{\underline{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}}.$$

Oppgave 4

a) $\det A = \begin{vmatrix} a & 1 & 0 \\ 2 & a & 2 \\ 0 & 1 & a \end{vmatrix} = a(a^2-2) - 2a = a(a^2-4).$

A er inverterbar $\Leftrightarrow \det A \neq 0 \Leftrightarrow \underline{\underline{a \neq 0, \pm 2}}.$

Med $a = -1$ får vi

$$\left[\begin{array}{rrr|rrr} -1 & 1 & 0 & 1 & 0 & 0 \\ 2 & -1 & 2 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{rrr|rrr} -1 & 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 2 & 2 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right] \sim$$

$$\left[\begin{array}{rrr|rrr} 1 & 0 & 2 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 1 & 0 \\ 0 & 0 & -3 & -2 & -1 & 1 \end{array}\right] \sim \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & -1/3 & 1/3 & 2/3 \\ 0 & 1 & 0 & 2/3 & 1/3 & 2/3 \\ 0 & 0 & 1 & 2/3 & 1/3 & -1/3 \end{array}\right]$$

Dvs. $\underline{\underline{A^{-1} = \frac{1}{3}\begin{bmatrix} -1 & 1 & 2 \\ 2 & 1 & 2 \\ 2 & 1 & -1 \end{bmatrix}}}$

b) Egenverdiene:

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 & 0 \\ 2 & 1-\lambda & 2 \\ 0 & 1 & 1-\lambda \end{vmatrix} = (1-\lambda)\left[(1-\lambda)^2 - 2\right] - 2(1-\lambda)$$

$$= -(\lambda - 1)(\lambda^2 - 2\lambda - 3) = -(\lambda+1)(\lambda-1)(\lambda-3) = 0$$

gir $\underline{\underline{\lambda_1 = -1,\ \lambda_2 = 1,\ \lambda_3 = 3}}$.

Egenvektorer:

$$\lambda_1 = -1 \quad \text{gir} \quad \begin{bmatrix} 2 & 1 & 0 \\ 2 & 2 & 2 \\ 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \qquad \Rightarrow \mathbf{x}_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix},$$

$$\lambda_1 = 1 \quad \text{gir} \quad \begin{bmatrix} 0 & 1 & 0 \\ 2 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \Rightarrow \mathbf{x}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix},$$

$$\lambda_1 = 3 \quad \text{gir} \quad \begin{bmatrix} -2 & 1 & 0 \\ 2 & -2 & 2 \\ 0 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \qquad \Rightarrow \mathbf{x}_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

Med disse egenvektorene kan vi sette

$$\underline{\underline{P = \begin{bmatrix} 1 & 1 & 1 \\ -2 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix}}} \quad \text{og} \quad \underline{\underline{D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}}}.$$

c) Fra b) får vi

$$\underline{\underline{\mathbf{y} = c_1 e^{-t}\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + c_2 e^{t}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c_3 e^{3t}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}}}.$$

Oppgave 5

Fra $P^T P = I$ får vi

$$1 = \det(P^T P) = \det P^T \det P = (\det P)^2,$$

så $\det P = \pm 1$.

Oppgave 6

Her er

$$y_1' = -\frac{6}{200}y_1 + \frac{2}{100}y_2$$
$$y_2' = \frac{2}{200}y_1 - \frac{2}{100}y_2,$$

dvs. $\mathbf{y}' = -\frac{1}{100}A\mathbf{y}$ der $A = \begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix}$.

A har karakteristisk likning $\lambda^2 - 5\lambda + 4 = 0$, dvs. egneverdiene er $\lambda_1 = 1$ og $\lambda_2 = 4$. Tilhørende egenvektorer er

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{og} \quad \mathbf{x}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix},$$

så generell løsning er

$$\mathbf{y} = c_1 e^{-\frac{t}{100}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-\frac{t}{25}} \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

Initialbetingelsene $y_1(0) = 1$ og $y_2(0) = 7$ gir

$$c_1 + 2c_2 = 1$$
$$c_1 - c_2 = 7,$$

dvs. $c_1 = 5, c_2 = -2$, så $\mathbf{y} = 5e^{-\frac{t}{100}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} - 2e^{-\frac{t}{25}} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, dvs.

$$y_1(t) = 5e^{-\frac{t}{100}} - 4e^{-\frac{t}{25}}$$
$$\underline{\underline{y_2(t) = 5e^{-\frac{t}{100}} + 2e^{-\frac{t}{25}}}}.$$

Vi har $y_2(T) = 2y_1(T)$ når

$$5e^{-\frac{T}{100}} + 2e^{-\frac{T}{25}} = 2\left(5e^{-\frac{T}{100}} - 4e^{-\frac{T}{25}}\right).$$

Dvs.

$$10e^{-\frac{4T}{100}} = 5e^{-\frac{T}{100}} \quad \Leftrightarrow \quad e^{-\frac{3T}{100}} = \frac{1}{2} \quad \Leftrightarrow \quad \underline{\underline{T = \frac{100}{3}\ln 2}} \text{ (sekunder)}$$