SECOND-ORDER EQUATIONS WITH CONSTANT COEFFICIENTS AND THE METHOD OF UNDETERMINED COEFFICIENTS

Given a second-order linear differential equation with **constant coefficients**

$$(*) \;\; y'' + ay' + by = r(x)$$

with corresponding **homogeneous** equation

$$(**) \;\; y'' + ay' + by = 0.$$

The **characteristic equation** is

$$(1) \;\; \lambda^2 + a\lambda + b = 0.$$

The **homogeneous** equation $(**)$ is solved by solving (1):

1) If (1) has two **simple** real roots (i.e. distinct real roots) λ_1 og λ_2, a general solution of $(**)$ is given by $y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$.
2) If (1) has a **double root** $\lambda_1 = \lambda_2 = \lambda$, a general solution of $(**)$ is given by $y = c_1 e^{\lambda x} + c_2 x e^{\lambda x}$.
3) If $\lambda_{1,2} = \alpha \pm i\beta$ $(\beta \neq 0)$ are **complex** roots of (1), a general solution of $(**)$ is given by $y = c_1 e^{\alpha x} \cos \beta x + c_2 e^{\alpha x} \sin \beta x$.

Below, you will find a table giving the form of a **particular solution** y_p of the **nonhomogeneous** equation $(*)$ for **certain** $r(x)$. Here $P_n(x)$ denotes a polynomial of degree $n \geq 0$, $P_n(x) = a_0 + a_1 x + \cdots + a_n x^n$ where $a_n \neq 0$. Note too that c and α are 0 when there are no exponential functions in $r(x)$.

$r(x)$	y_p
a) $P_n(x)e^{cx}$	$x^m(A_0 + A_1 x + \cdots + A_n x^n)e^{cx}$ where $m = \begin{cases} 0 \text{ if } c \text{ is **not** a root of (1)} \\ 1 \text{ if } c \text{ is a **simple** root of (1)} \\ 2 \text{ if } c \text{ is a **double** root of (1).} \end{cases}$
b) $P_n(x)e^{\alpha x}\cos\beta x$ or $P_n(x)e^{\alpha x}\sin\beta x$	$x^m[(A_0 + A_1 x + \cdots + A_n x^n)e^{\alpha x}\cos\beta x + (B_0 + B_1 x + \cdots + B_n x^n)e^{\alpha x}\sin\beta x]$ where $m = \begin{cases} 0 \text{ if } \alpha \pm i\beta \text{ is **not** a root of (1)} \\ 1 \text{ if } \alpha \pm i\beta \text{ **is** a root of (1).} \end{cases}$
If $r(x)$ is a sum of terms as in a) og b), the choice of y_p is a sum of the corresponding terms.	