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1.5 Solution Sets of Linear Systems

In Exercise 1 and 3, determine if the system has a nontrivial solution. Try to use as few row operations as possible.

1.

$$\begin{aligned} 2x_1 - 5x_2 + 8x_3 &= 0 \\ -2x_1 - 7x_2 + x_3 &= 0 \\ 4x_1 + 2x_2 + 7x_3 &= 0 \end{aligned}$$

3.

$$\begin{aligned} -3x_1 + 5x_2 - 7x_3 &= 0 \\ -6x_1 + 7x_2 + x_3 &= 0 \end{aligned}$$

5. Follow the method of Examples 1 and 2 to write the solutions set of the given homogeneous system in parametric vector form.

$$\begin{aligned} x_1 + 3x_2 + x_3 &= 0 \\ -4x_1 - 9x_2 + 2x_3 &= 0 \\ -3x_2 - 6x_3 &= 0 \end{aligned}$$

In Exercise 7 and 10, describe all solutions of $A\mathbf{x} = \mathbf{0}$ in parametric vector form, where A is row equivalent to the given matrix.

7.

$$\begin{bmatrix} 1 & 3 & -3 & 7 \\ 0 & 1 & -4 & 5 \end{bmatrix}$$

10.

$$\begin{bmatrix} 1 & 3 & 0 & -4 \\ 2 & 6 & 0 & -8 \end{bmatrix}$$

23. Mark each statement True or False. Justify each answer.

a. A homogeneous equation is always consistent.

b. The equation $A\mathbf{x} = \mathbf{0}$ gives an explicit description of its solution set.

c. The homogeneous equation $A\mathbf{x} = \mathbf{0}$ has the trivial solution if and only if the equation has at least one free variable.

d. The equation $\mathbf{x} = \mathbf{p} + t\mathbf{v}$ describes a line through $\mathbf{v}$ parallel to $\mathbf{p}$.

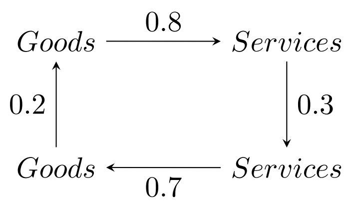
e. The solution set of $A\mathbf{x} = \mathbf{b}$ is the set of all vectors of the form $\mathbf{w} = \mathbf{p} + \mathbf{v}_h$, where $\mathbf{v}_h$ is any solution of the equation $A\mathbf{x} = \mathbf{0}$.

35. Construct a 3×3 nonzero matrix A such that the vector $[1, 1, 1]^T$ is a solution of $A\mathbf{x} = \mathbf{0}$.

39. (Optional Extra) Let A be an $m \times n$ matrix and let $\mathbf{u}$ be a vector in $\mathbb{R}^n$ that satisfies the equation $A\mathbf{x} = \mathbf{0}$. Show that for any scalar c, the vector $c\mathbf{u}$ also satisfies $A\mathbf{x} = \mathbf{0}$. (That is, show that $A(c\mathbf{u}) = \mathbf{0}$).

1.6 Applications of Linear Systems

1. Suppose an economy has only two sectors, Goods and Services. Each year, Goods sells 80% of its output to Services and keeps the rest, while Services sells 70% of its output to Goods and retains the rest. Find equilibrium prices for the annual outputs of the Goods and Services sectors that make each sector's income match its expenditures. (See the figure in the textbook page 71).



7. Alka-Seltzer contains sodium bicarbonate ($NaHCO_3$) and citric acid ($H_3C_6H_5O_7$). When a tablet is Dissolved in water, the following reaction produces sodium citrate, water, and carbon dioxide (gas):

$$NaHCO_3 + H_3C_6H_5O_7 \rightarrow Na_3C_6H_5O_7 + H_2O + CO_2$$

Balance the chemical equation using the vector equation approach discussed in this section.

1.7 Linear Independence

1. Determine if the vectors are linearly independent. Justify the answer.

$$\begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 7 \\ 2 \\ -6 \end{bmatrix}, \begin{bmatrix} 9 \\ 4 \\ -8 \end{bmatrix}$$

5. Determine if the columns of the matrix form a linearly independent set. Justify the answer.

$$\begin{bmatrix} 0 & -8 & 5 \\ 3 & -7 & 4 \\ -1 & 5 & -4 \\ 1 & -3 & 2 \end{bmatrix}$$

13. Find the value(s) of h for which the vectors are linearly *dependent*. Justify the answer.

$$\begin{bmatrix} 1 \\ 5 \\ -3 \end{bmatrix}, \begin{bmatrix} -2 \\ -9 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ h \\ -9 \end{bmatrix}$$

27. How many pivot columns must a 7×5 matrix have if its columns are linearly independent? Why?

39. (Optional Extra) Suppose A is an $m \times n$ matrix with the property that for all $\mathbf{b}$ in $\mathbb{R}^m$ the equation $A\mathbf{x} = \mathbf{b}$ has at most one solution. Use the definition of linear independence to explain why the columns of A must be linearly independent.