TMA4110 Calculus 3
Fall 2016

Øving 7

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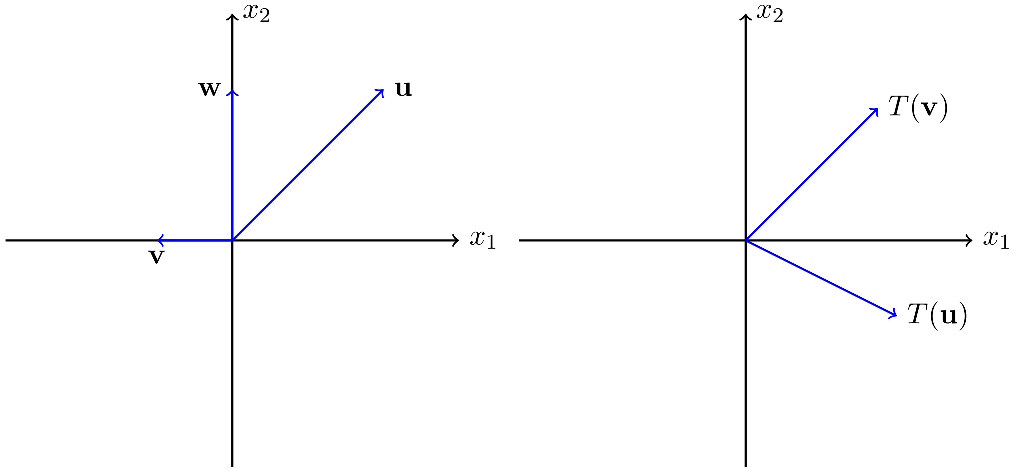
1.8 Introduction to Linear Transformations

8. How many rows and columns must a matrix A have in order to define a mapping from $\mathbb{R}^4$ into $\mathbb{R}^5$ by the rule $T(\mathbf{x}) = A\mathbf{x}$?

17. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation that maps $\mathbf{u} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$ into $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and maps $\mathbf{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ into $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$. Use the fact that T is linear to find the images under T of $3\mathbf{u}$, $2\mathbf{v}$, and $3\mathbf{u} + 2\mathbf{v}$.

18. The figure shows vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$, along with the images $T(\mathbf{u})$ and $T(\mathbf{v})$ under the action of a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$. Copy this figure carefully, and draw the image $T(\mathbf{w})$ as accurately as possible. (*Hint:* First, write $\mathbf{w}$ as a linear combination of $\mathbf{u}$ and $\mathbf{v}$.)



25. (Optional Extra) Given $\mathbf{v} \neq \mathbf{0}$ and $\mathbf{p}$ in $\mathbb{R}^n$, the line through $\mathbf{p}$ in the direction of $\mathbf{v}$ has the parametric equation $\mathbf{x} = \mathbf{p} + t\mathbf{v}$. Show that a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ maps this line onto another line or onto a single point (a *degenerate line*).

26. (Optional Extra) Let $\mathbf{u}$ and $\mathbf{v}$ be linearly independent vectors in $\mathbb{R}^3$, and let P be the plane through $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{0}$. The parametric equation of P is $\mathbf{x} = s\mathbf{u} + t\mathbf{v}$ (with s, t in $\mathbb{R}$). Show that a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ maps P onto a plane through $\mathbf{0}$, or onto a line through $\mathbf{0}$, or onto just the origin in $\mathbf{R}^3$. What must be true about $T(\mathbf{u})$ and $T(\mathbf{v})$ in order for the image of the plane P to be a plane?

30. (Optional Extra) An *affine transformation* $T : \mathbb{R}^n \to \mathbb{R}^m$ has the form $T(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$, with A an $m \times n$ matrix and $\mathbf{b}$ in $\mathbb{R}^m$. Show that T is *not* a linear transformation when $\mathbf{b} \neq \mathbf{0}$. (Affine transformation are important in computer graphics.)

31. (Optional Extra) Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, and let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a linearly dependent set in $\mathbb{R}^n$. Explain why the set $\{T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)\}$ is linearly dependent.

1.9 The Matrix of a Linear Transformation

In Exercise **1**, **5**, and **8**, assume that T is a linear transformation. Find the standard matrix of T.

1. $T : \mathbb{R}^2 \to \mathbb{R}^4$, $T(\mathbf{e}_1) = (3, 1, 3, 1)$ and $T(\mathbf{e}_2) = (-5, 2, 0, 0)$, where $\mathbf{e}_1 = (1, 0)$ and $\mathbf{e}_2 = (0, 1)$.

5. $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a vertical shear transformation that maps $\mathbf{e}_1$ into $\mathbf{e}_1 - 2\mathbf{e}_2$ but leaves the vector $\mathbf{e}_2$ unchanged.

8. $T : \mathbb{R}^2 \to \mathbb{R}^2$ first reflects points through the horizontal x_1-axis and then reflects points through the line $x_2 = x_1$.

12. (Optional Extra) Show that the transformation in Exercise **8.** is merely a rotation about the origin. What is the angle of the rotation?

15. Fill in the missing entries of the matrix, assuming that the equation holds for all values of the variables.

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3x_1 - 2x_3 \\ 4x_1 \\ x_1 - x_2 + x_3 \end{bmatrix}$$

(Optional Extra) In Exercise **29** and **30**, describe the possible echelon forms of the standard matrix for a linear transformation T. Use the notation of Example 1 in Section 1.2.

29. (Optional Extra) $T : \mathbb{R}^3 \to \mathbb{R}^4$ is one-to-one.

30. (Optional Extra) $T : \mathbb{R}^4 \to \mathbb{R}^3$ is onto.

31. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, with A its standard matrix. Complete the following statement to make it true: "T is one-to-one if and only if A has _ pivot columns." Explain why the statement is true. (*Hint:* Look in the exercises for Section 1.7.)

32. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, with A its standard matrix. Complete the following statement to make it true: "T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if A has _ pivot columns." Find some theorems that explain why the statement is true.

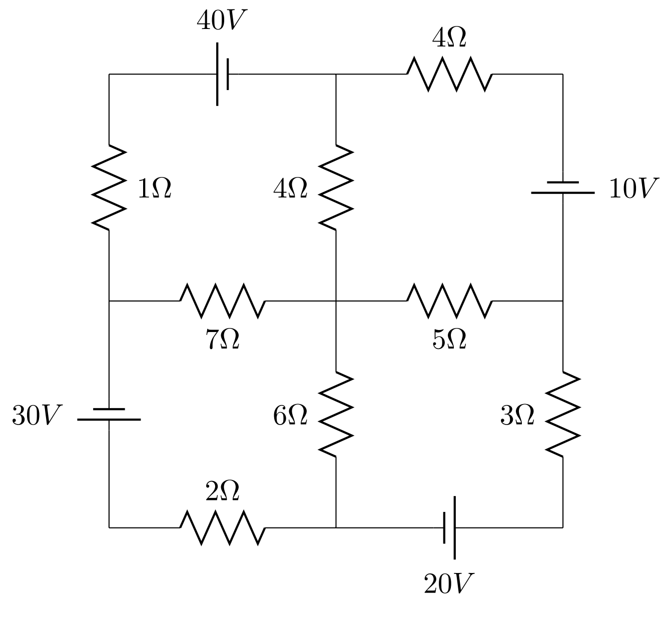
35. If a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ maps $\mathbb{R}^n$ onto $\mathbb{R}^m$, can you give a relation between m and n? If T is one-to-one, what can you say about m and n?

36. (Optional Extra) Let $S : \mathbb{R}^p \to \mathbb{R}^n$ and $T : \mathbb{R}^n \to \mathbb{R}^m$ be linear transformations. Show that the mapping $\mathbf{x} \mapsto T(S(\mathbf{x}))$ is a linear transformation (from $\mathbb{R}^p$ to $\mathbb{R}^m$). (*Hint:* Compute $T(S(c\mathbf{u} + d\mathbf{v}))$ for $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^p$ and scalars c and d. Justify each step of the computation, and explain why this computation gives the desired conclusion.)

1.10 Linear Models in Business, Science, and Engineering

4a. The Cambridge Diet supplies $0.8g$ of calcium per day, in addition to the nutrients listed in Table 1 of Example 1. The amounts of calcium per unit ($100g$) supplied by the three ingredients in the Cambridge Diet are as follows: $1.26g$ from nonfat milk, $0.19g$ from soy flour, and $0.8g$ from whey. Another ingredient in the diet mixture is isolated soy protein, which provides the following nutrients in each unit: $80g$ of protein, $0g$ of carbohydrate, $3.4g$ of fat, and $0.18g$ of calcium.
Set up a matrix equation whose solution determines the amounts of nonfat milk, soy flour, whey, and isolated soy protein necessary to supply the precise amount of protein, carbohydrate, fat, and calcium in the Cambridge Diet. State what the variables in the equation represent.

7. (Optional Extra) Write a matrix equation that determines the loop currents.



11a. In 2012 the population of California was $38,041,430$, and the population living in the United States but *outside* California wa $275,872,610$. During the year, is is estimated that $748,252$ persons moved from California to elsewhere in the United States, while $493,641$ persons moved to California from elsewhere in the United States.
Set up the migration matrix for this situation, using five decimal places for the migration rates into and out of California. Let your work show how you produced the migration matrix.