Introductory course in linear algebra and differential equations
Autumn 2019

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Solutions to exercise set 2

$\boxed{1}$ **a)** We do Gauss elimination (we don't augment the matrix, since there are only 0's on the right hand side) and find the reduced echelon form

$$\begin{bmatrix}1 & 1 & 3 & 2\\ 3 & -2 & -1 & -4\\ 4 & 1 & 6 & 2\end{bmatrix} \sim \begin{bmatrix}1 & 1 & 3 & 2\\ 0 & -5 & -10 & -10\\ 0 & -3 & -6 & -6\end{bmatrix}$$
$$\sim \begin{bmatrix}1 & 1 & 3 & 2\\ 0 & 1 & 2 & 2\\ 0 & 0 & 0 & 0\end{bmatrix}$$
$$\sim \begin{bmatrix}1 & 0 & 1 & 0\\ 0 & 1 & 2 & 2\\ 0 & 0 & 0 & 0\end{bmatrix}.$$

Here $x_3 = s$ and $x_4 = t$ are free variables. We get $x_1 = -s$ og $x_2 = -2s - 2t$. The solutions are thus given by

$$x = (-s, -2s - 2t, s, t) = s(-1, -2, 1, 0) + t(0, -2, 0, 1)$$

for $s, t \in \mathbb{R}$.

b) From the previous exercise we get that a basis for Row(A) is

$$\{(1, 0, 1, 0), (0, 1, 2, 2)\}$$

and that a basis for Nul(A) is

$$\{(-1, -2, 1, 0), (0, -2, 0, 1)\}.$$

Since the two first columns in the echelon form are pivot columns, then

$$\{(1, 3, 4), (1, -2, 1)\}$$

is basis for Col(A).
Finally, we have that $\mathrm{rank}(A) = \dim(\mathrm{Row(A)}) = \dim(\mathrm{Col(A)}) = 2$.

$\boxed{2}$ **a)** The Matrix A is invertible whenever it has maximum rank (which is 3). By Gaussian elimination we find

$$\begin{bmatrix}1 & 0 & a\\ a & 1 & 0\\ 0 & a & 1\end{bmatrix} \sim \begin{bmatrix}1 & 0 & a\\ 0 & 1 & -a^2\\ 0 & a & 1\end{bmatrix}$$
$$\sim \begin{bmatrix}1 & 0 & a\\ 0 & 1 & -a^2\\ 0 & 0 & 1 + a^3\end{bmatrix}.$$

From the echelon form we see that A has maximum rank if and only if $1+a^3 \neq 0$, thus whenever $a \neq -1$. Thus the matrix is invertible if and only if $a \neq -1$.

b) Multiply the two matrices and check that you get the identity matrix (it is enough to check one of the two products).

$\boxed{3}$ **a)** It is clear that $0 \in Q$ (the polynomial $p = 0$ is zero everywhere!). Let now $\alpha, \beta \in \mathbb{R}$ and $p, q \in Q$. Then

$$(\alpha p + \beta q)(0) = \alpha p(0) + \beta q(0) = \alpha \cdot 0 + \beta \cdot 0 = 0,$$

such that $\alpha p + \beta q \in Q$. This means that Q is closed under addition and multiplication by a constant. Thus Q is a subspace of P_4.

b) We have that $p(0) = 0$ if and only if x is a factor of p. This means that Q consists of polynomials on the form

$$\begin{aligned} p(x) &= x(a_1 + a_2 x + a_3 x^2) \\ &= a_1 x + a_2 x^2 + a_3 x^3. \end{aligned}$$

A basis for Q is $\{x, x^2, x^3\}$, and $\dim(Q) = 3$.