Norwegian University of Science and Technology
Department of Mathematical Sciences

Introductory course in linear algebra and differential equations Autumn 2019

Solutions to exercise set 3

$\boxed{1}$ **a)** We calculate,

$$\begin{vmatrix} 3 & 4 \\ 2 & 5 \end{vmatrix} = 3 \cdot 5 - 2 \cdot 4 = 15 - 8 = 7.$$

This matrix is invertible, since the deterimant is non-zero.

b) We start with the first column (since there is a 0 there)

$$\begin{vmatrix} -2 & 2 & 2 \\ 3 & 3 & 1 \\ 0 & 1 & -2 \end{vmatrix} = (-2)\begin{vmatrix} 3 & 1 \\ 1 & -2 \end{vmatrix} - 3\begin{bmatrix} 2 & 2 \\ 1 & -2 \end{bmatrix} = (-2)(-7) - 3(-6) = 32$$

Invertible.

c) The last two rows of the matrix are linearly dependent. Thus the matrix has not maximum rank, and is thus not invertible. The determinant is zero. (Alternatively, the determinant of an upper triangular matrix is the product of the numbers on the diagonal. Here 0 lies on the diagonal, and thus the product is 0.)

$\boxed{2}$ We start with the first row

$$\det(A) = \begin{vmatrix} 1 & 0 \\ a & 1 \end{vmatrix} + a\begin{vmatrix} a & 1 \\ 0 & a \end{vmatrix} = 1 + a^3.$$

The matrix is invertible if and only if $a \neq -1$. This is the same as we found yesterday.

$\boxed{3}$ We calculate the polynomial,

$$p(\lambda) = \det(U - \lambda I) = \begin{vmatrix} a - \lambda & b \\ 0 & c - \lambda \end{vmatrix} = (a - \lambda)(c - \lambda).$$

The eigenvalues are a and c.

For the general case (U an $n \times n$ upper triangular matrix):

$$p(\lambda) = \det(U - \lambda I) = \begin{vmatrix} a_{11} - \lambda & \cdots & & \\ 0 & a_{22} - \lambda & & \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} - \lambda \end{vmatrix} = (a_{11} - \lambda)(a_{22} - \lambda) \cdots (a_{nn} - \lambda).$$

The eigenvalues are simply the numbers on the diagonal!

$\boxed{4}$ **a)** The characteristic polynomial is

$$p(\lambda) = (5-\lambda)(-\lambda) + 6 = \lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3).$$

The eigenvalues are $\lambda_1 = 2$ og $\lambda_2 = 3$, both with algebraic multiplicity 1. The geometric multiplicities are also 1 (since they are greater or equal to 1, and less or equal the algebraic multiplicity)
Since

$$A - 2 \cdot I = \begin{bmatrix} 3 & 2 \\ -3 & -2 \end{bmatrix} \sim \begin{bmatrix} 3 & 2 \\ 0 & 0 \end{bmatrix}$$

the eigenvalue λ_1 corresponds to the eigenvector $v_1 = (2, -3)$. Furthermore,

$$A - 3 \cdot I = \begin{bmatrix} 2 & 2 \\ -3 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix},$$

so that for λ_2 the eigenvector is $v_2 = (1, -1)$.

b) The characteristic polynomial is

$$p(\lambda) = (1-\lambda) \begin{vmatrix} -2-\lambda & 1 \\ -9 & 4-\lambda \end{vmatrix} = (1-\lambda)((-2-\lambda)(4-\lambda) + 9) = (1-\lambda)^3,$$

so that the only eigenvalue is $\lambda_1 = 1$, with algebraic multiplicity 3. Since

$$B - I = \begin{bmatrix} 0 & -1 & 0 \\ 0 & -3 & 1 \\ 0 & -9 & 3 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

there is only one linearly independent eigenvector corresponding to this eigenvalue, namely $v_1 = (1, 0, 0)$. It follows that the geometric multiplicity of λ_1 equals 1.