Norwegian University of Science and Technology
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Introductory course in linear algebra and differential equations
Autumn 2019

Solutions to exercise set 4

1 We denote by c_n and s_n the number of people in the centre and the number of people in the suburbs after n years, correspondingly. At year 0, we have $c_0 = 7$ og $s_0 = 5$. From the given information, we get

$$\begin{bmatrix} c_n \\ s_n \end{bmatrix} = \begin{bmatrix} 0.8c_{n-1} + 0.1s_{n-1} \\ 0.2c_{n-1} + 0.9s_{n-1} \end{bmatrix} = \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix} \begin{bmatrix} c_{n-1} \\ s_{n-1} \end{bmatrix}, = A \begin{bmatrix} c_{n-1} \\ s_{n-1} \end{bmatrix}.$$

By iteration we get

$$\begin{bmatrix} c_n \\ s_n \end{bmatrix} A^2 \begin{bmatrix} c_{n-2} \\ s_{n-2} \end{bmatrix} = \cdots = A^n \begin{bmatrix} c_0 \\ s_0 \end{bmatrix} = PD^nP^{-1} \begin{bmatrix} c_0 \\ s_0 \end{bmatrix}.$$

Thus we get,

$$\begin{aligned} \begin{bmatrix} c_n \\ s_n \end{bmatrix} &= \frac{1}{3} \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0.7^n \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 7 \\ 5 \end{bmatrix} \\ &= \frac{1}{3} \begin{bmatrix} 1 & -0.7^n \\ 2 & 0.7^n \end{bmatrix} \begin{bmatrix} 12 \\ -9 \end{bmatrix} \\ &= \begin{bmatrix} 4 + 3 \cdot 0.7^n \\ 8 - 3 \cdot 0.7^n \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix} + 0.7^n \begin{bmatrix} 3 \\ -3 \end{bmatrix}. \end{aligned}$$

From this we see that $(c_n, s_n) \to (4, 8)$ as $n \to \infty$ (because $0.7 < 1$). In the long term, the number of people in the centre will be 4 million, and the number of people in the suburbs will be 8 million.

2 **a)** We expand along the second row/column and get

$$\begin{aligned} p(\lambda) &= (4-\lambda) \begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} \\ &= (4-\lambda)(\lambda^2 - 4\lambda + 3) \\ &= (4-\lambda)(\lambda - 1)(\lambda - 3). \end{aligned}$$

b) From the characteristic polynomial we see that A has the eigenvalues $\lambda_1 = 1$, $\lambda_2 = 3$ and $\lambda_3 = 4$. Since

$$A - I = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 3 & 0 \\ -1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

λ_1 has the eigenvector $w_1 = (1, 0, 1)$. Further, we get

$$A - 3I = \begin{bmatrix} -1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

so that $w_2 = (1, 0, -1)$ is an eigenvector corresponding to λ_2. Finally for λ_3

$$A - 4I = \begin{bmatrix} -2 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

the corresponding eigenvector is $w_3 = (0, 1, 0)$.

c) The vectors w_1, w_2, w_3 are orthogonal (as they should be, since A is symmetric, and they correspond to distinct eigenvalues). Since $|w_1| = |w_2| = \sqrt{2}$ and $|w_3| = 1$ we define

$$\begin{aligned} v_1 &= w_1/\sqrt{2} = (1/\sqrt{2}, 0, 1/\sqrt{2}) \\ v_2 &= w_2/\sqrt{2} = (1/\sqrt{2}, 0, -1/\sqrt{2}) \\ v_3 &= w_1 = (0, 1, 0), \end{aligned}$$

which is then an orthonormal basis for $\mathbb{R}^3$ consisting of eigenvectors of A. By defining the matrix

$$P = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{bmatrix}$$

and

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix},$$

we get that

$$\begin{aligned} A &= PDP^T \\ &= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{bmatrix} \end{aligned}$$

is an orthogonal diagonalization of A.

d) Vi calculate

$$\begin{aligned} e^A &= Pe^D P^T \\ &= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} e & 0 & 0 \\ 0 & e^3 & 0 \\ 0 & 0 & e^4 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}}e & 0 & \frac{1}{\sqrt{2}}e \\ \frac{1}{\sqrt{2}}e^3 & 0 & -\frac{1}{\sqrt{2}}e^3 \\ 0 & e^4 & 0 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2}e + \frac{1}{2}e^3 & 0 & \frac{1}{2}e - \frac{1}{2}e^3 \\ 0 & e^4 & 0 \\ \frac{1}{2}e - \frac{1}{2}e^3 & 0 & \frac{1}{2}e + \frac{1}{2}e^3 \end{bmatrix}. \end{aligned}$$