Norwegian University of Science and Technology
Department of Mathematical Sciences

Introductory course in linear algebra and differential equations
Autumn 2019

Solutions to exercise set 5

$\boxed{1}$ By differentiation, we get that

$$y'(t) = \frac{1}{1+t^2}, \quad y''(t) = -\frac{2t}{(1+t^2)^2},$$

so if y solves the differential equation, we get that

$$(t^2+1)\left(\frac{-2t}{(1+t^2)^2}\right) + at\frac{1}{1+t^2} = 0.$$

Thus,

$$\frac{-2t+at}{1+t^2} = 0,$$

so that for $t \neq 0$ we get $a = 2$.

$\boxed{2}$ **a)** We choose $x_1 = y'$ and $x_2 = y$. Then we get

$$\begin{aligned} \dot{x}_1 = y'' &= ay' + by + c \\ &= ax_1 + bx_2 + c \end{aligned}$$

and

$$\dot{x}_2 = y' = x_1.$$

We can thereby rewrite the differential equation as a system

$$\dot{x} = Ax + d,$$

where

$$x = (x_1, x_2), \quad A = \begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix}, \quad d = (c, 0).$$

b) We choose $x_1(t) = y(t)$ and $x_2(t) = t$. Then, we have

$$\begin{aligned} &\dot{x}_1 = y' = g(t)y = g(x_2)x_1 \\ &\dot{x}_2 = 1, \end{aligned}$$

so that we can rewrite the differential equation as

$$(\dot{x}_1, \dot{x}_2) = (g(x_2)x_1, 1)$$

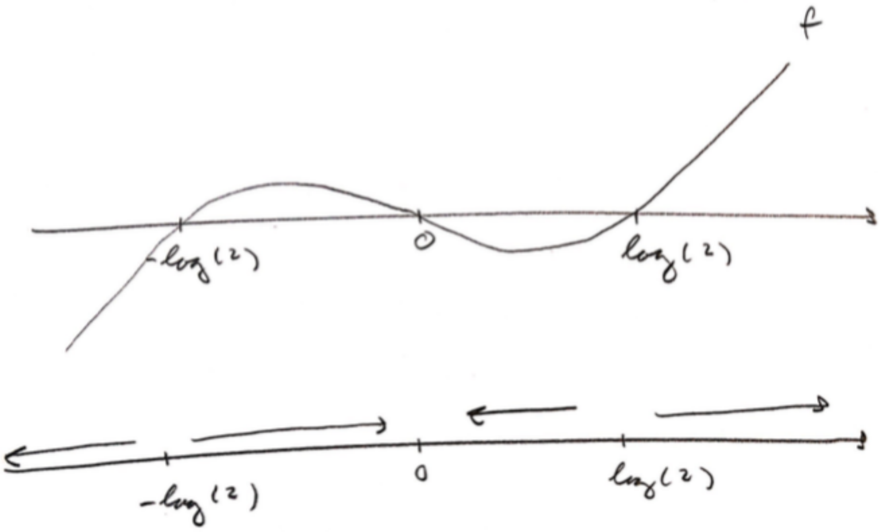
c) We choose $x_1 = y^{(3)}$, $x_2 = y''$, $x_3 = y'$ and $x_4 = y$. Then we have

$$\begin{aligned}
\dot{x}_1 &= y^{(4)} = \sin(y)(y')^2 = \sin(x_4)x_3^2 \\
\dot{x}_2 &= y^{(3)} = x_1 \\
\dot{x}_3 &= x_2 \\
\dot{x}_4 &= x_3,
\end{aligned}$$

so that we can write the differential equation on the form $\dot{x} = f(x)$ with $x = (x_1, \ldots, x_4)$

$$f(x) = (\sin(x_4)x_3^2, x_1, x_2, x_3).$$

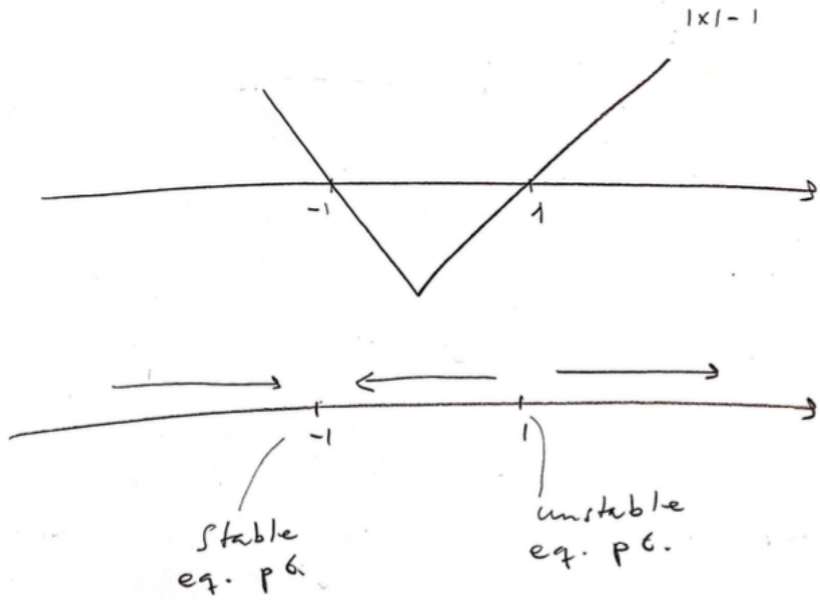
$\boxed{3}$ **a)** The function f is zero in the points $0, \pm\log(2)$. These are the equilibrium points of the differential equation. Below is a drawing of f, and the qualitative behaviour of the solutions to the differential equation.

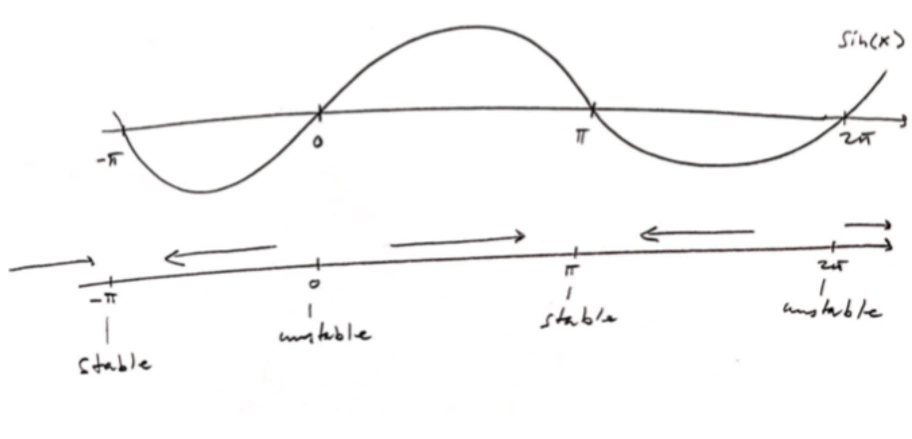


- If our initial data is on the interval $(-\infty, -\log(2))$ the solution will go to $-\infty$ as $t \to \infty$.
- On the interval $(-\log(2), 0)$ the solution tends to the equilibrium point 0.
- On the interval $(0, \log(2), 0)$ the solution also tends to the equilibrium point 0.
- On the interval $(\log(2), \infty)$ the solutions tends to $+\infty$.

From the list we observe that the equilibrium point 0 is stable, while $\pm\log(2)$ are unstable.

b) and c) We make the following drawings:


|x| - 1
-1
1
-1
1
stable
eq. pt.
unstable
eq. pt.


sin(x)
-π
0
π
2π
-π
stable
0
unstable
π
stable
2π
unstable